Saginaw Valley State University

2007 Math Olympics – Level II Solutions

1. For which x is $\log_2(x^2 - 2x - 2) \le 0$?

(a) $(-1, 3)$ **(b)** $[-1, 3]$

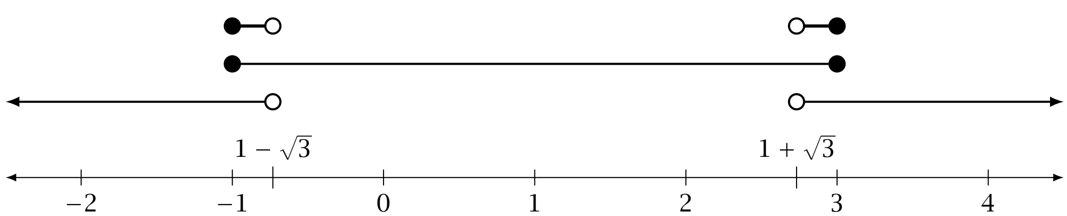
(c) $(-\infty, 1-\sqrt{3}) \cup (1+\sqrt{3}, \infty)$ **(d)** $[-1, 1-\sqrt{3}) \cup (1+\sqrt{3}, 3]$

(e) None of the above

Solution **(d)**: First of all, in order for the $\log_2$ to be even defined, $x^2 - 2x - 2 > 0$. The parabola with equation $y = x^2 - 2x - 2$ opens up and has for x-intercepts $x = 1 \pm \sqrt{3}$ (for example using quadratic formula). The parabola will be above the x-axis if $x < 1 - \sqrt{3}$ or if $x > 1 + \sqrt{3}$.

The logarithm will be less than or equal to 0 if $x^2 - 2x - 2 \le 1$, or $x^2 - 2x - 3 \le 0$. The parabola with equation $y = x^2 - 2x - 3$ opens up and has x-intercepts for $x = -1$ and $x = 3$. Therefore $x^2 - 2x - 3 \le 0$ if $-1 \le x \le 3$.

So x must be in the interval $[-1, 3]$ but outside of the interval $[1-\sqrt{3}, 1+\sqrt{3}]$:



2. The exact value of the expression $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is

(a) $\frac{2\pi}{3}$ **(b)** $\frac{\sqrt{3}}{2}$ **(c)** $\frac{\pi}{3}$ **(d)** $\sqrt{3}$ **(e)** None of the above

Solution **(c)**: The range of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The angle $\frac{2\pi}{3}$ is in the second quadrant, sin is positive, and the corresponding reference angle is $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

3. Augustus, Benedict, Claudio, and Diana have been accused of stealing the golden mean. It is known that one of these four people must have done it. Augustus says “Benedict did it”. Benedict says “Diana did it”. Claudio says “I didn’t do it”. Diana says “Benedict is lying when he says I did it”. If it is known that exactly one of them is lying, which one did it?

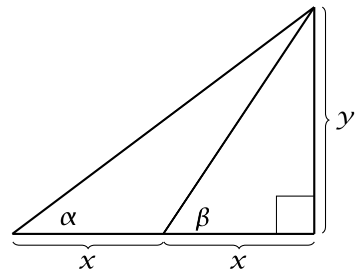
(a) Augustus **(b)** Benedict

(c) Claudio **(d)** Diana

(e) More than one person must be lying

Solution **(b)**: Since Benedict and Diana contradict each other, one of them must be lying. We know that only one person is lying, therefore Augustus (and Claudio, but that is not important) must be telling the truth.

4. Which of the following equations expresses the correct relationship between the angles α and β as defined in the figure?



(a) $\beta = \frac{1}{2}\sin^{-1}\left(\frac{1-5\sin^2(\alpha)}{1+3\sin^2(\alpha)}\right)$

(b) $\beta = \frac{1}{2}\cos^{-1}\left(\frac{1-5\cos^2(\alpha)}{1+3\cos^2(\alpha)}\right)$ **(c)** $\beta = \frac{1}{2}\cos^{-1}\left(\frac{1-5\sin^2(\alpha)}{1+3\sin^2(\alpha)}\right)$

(d) $\beta = \frac{1}{2}\sin^{-1}\left(\frac{1-5\cos^2(\alpha)}{1+3\cos^2(\alpha)}\right)$ **(e)** None of the above

Solution **(c)**: Without loss of generality we can assume that the hypotenuse of the larger triangle is 1 unit long. Then $y = sin\alpha$ and $x = \frac{1}{2}\cos\alpha$. The hypotenuse of the short triangle is then

$$\sqrt{\frac{\cos^2\alpha}{4} + \sin^2\alpha} = \frac{1}{2}\sqrt{\cos^2\alpha + 4\sin^2\alpha} = \frac{1}{2}\sqrt{1+3\sin^2\alpha}$$

Then

$$\cos\beta = \frac{\cos\alpha}{\sqrt{1+3\sin^2\alpha}}$$

and

$$\cos^2\beta = \frac{\cos^2\alpha}{1+3\sin^2\alpha} = \frac{1-\sin^2\alpha}{1+3\sin^2\alpha}$$

Finally

$$\cos 2\beta = 2\cos^2\beta - 1 = \frac{1-\sin^2\alpha}{1+3\sin^2\alpha} - 1 = \frac{1-5\sin^2\alpha}{1+3\sin^2\alpha}$$

Applying inverse cosine and dividing by 2 we get

$$\beta = \frac{1}{2}\cos^{-1}\left(\frac{1-5\sin^2\alpha}{1+3\sin^2\alpha}\right)$$

5. The number of solutions to the equation $2^x + 4\left(2^{-x}\right) = 5$ is

(a) 0 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** More than 3

Solution **(c)**: Substituting $u = 2^x$ the equation becomes

$$\begin{aligned} u + \frac{4}{u} &= 5 \\ u^2 + 4 &= 5u \text{ (since } u = 2^x \text{ is never o)} \\ u^2 - 5u + 4 &= 0 \\ (u-4)(u-1) &= 0 \end{aligned}$$

therefore $u = 4$ and $x = 2$ or $u = 1$ and $x = 0$. Therefore the equation has 2 solutions.

6. The expression $\frac{1+\cos\theta}{\sin\theta} + \frac{\sin\theta}{1+\cos\theta}$ is equivalent to which of the following? (Assume $\theta \neq \pi + 2\pi n$ for any integer n.)

   (a) $2\csc\theta$ **(b)** $2\cot\theta$ **(c)** $2\sec\theta$ **(d)** $\csc\theta\sec\theta$

   (e) None of the above

   SOLUTION **(a)**:

$$\frac{1+\cos\theta}{\sin\theta} + \frac{\sin\theta}{1+\cos\theta} = \frac{(1+\cos\theta)^2+\sin^2\theta}{\sin\theta(1+\cos\theta)} = \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = 2\csc\theta$$

7. 10% of a high school senior class participate in Math Olympics. 95% of the seniors that participate in Math Olympics get into the college of their choice. Only 50% of the seniors who don't participate in Math Olympics get into the college of their choice. If a senior from that high school got into the college of his or her choice, what is the probability that this senior participated in Math Olympics?

   (a) .095 **(b)** .545 **(c)** $\frac{19}{109}$ **(d)** .95 **(e)** None of the above

   SOLUTION **(c)**: The probability that a student got into the college of his or her choice and participated in Math Olympics is $.1 \cdot .95 = .095$. the probability that a student got into a college of his or her choice and did not participate in Math Olympics is $.9 \cdot .5 = .45$. the probability that a student got into a college of his or her choice, regardless of participation in Math Olympics is the $.095 + .45 = .545$. The conditional probability that a student participated in Math Olympics provided he or she got into a college of his or her choice is

$$\frac{.095}{.545} = \frac{95}{545} = \frac{19}{109}$$

8. Let F be a function such that $F\left(\frac{8}{\sqrt{1+\sqrt{x}}}\right) = x$ for all $x \geq 0$. What is $F(4)$?

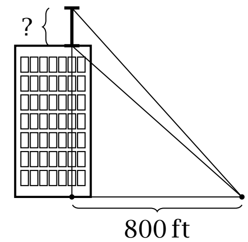
   (a) 9 **(b)** $\frac{8\sqrt{3}}{3}$ **(c)** $\frac{-12+8\sqrt{3}}{3}$ **(d)** $\frac{9}{16}$ **(e)** None of the above

   SOLUTION **(a)**: We are looking for x such that $4 = \frac{8}{\sqrt{1+\sqrt{x}}}$. Then $F(4) = x$.

$$\begin{aligned} 4 &= \frac{8}{\sqrt{1+\sqrt{x}}} \\ \sqrt{1+\sqrt{x}} &= 2 \\ 1+\sqrt{x} &= 4 \\ \sqrt{x} &= 3 \\ x &= 9 \end{aligned}$$

   Therefore $F(4) = 9$.

9. The largest advertising sign in the world is a large capital "I" on top of a building in L.A. At a distance of 800 ft along the ground from a point directly below the sign, the angle of elevation to the top of the sign is 45°. From this same point, the angle of elevation to the bottom of the sign is 30° . What is the height of the sign?



(a) 800 ft **(b)** $\frac{800\sqrt{3}}{3}$ ft

(c) $\frac{800(3-\sqrt{3})}{3}$ ft **(d)** $800\left(\sqrt{3}-1\right)$ ft **(e)** None of the above

Solution **(c)**: Let h be the height of the building itself and let x be the height of the sign. Then

$$\frac{h}{800} = \tan 30^\circ = \frac{\sqrt{3}}{3}$$

and

$$\frac{x+h}{800} = \tan(30^\circ + 15^\circ) = \tan 45^\circ = 1$$

Therefore

$$\frac{\sqrt{3}}{3} + \frac{x}{800} = 1$$

and

$$x = 800\left(\frac{3-\sqrt{3}}{3}\right)$$

10. A middle school has 100 lockers numbered 1 to 100, and 100 students. The first student goes down the row of lockers and opens every locker. Then the second student goes down the row of lockers and closes every locker that is numbered with a multiple of two. Then the third student goes down the row of lockers, and for every locker that is numbered with a multiple of 3, if it is open, she closes it, but if it is already closed, she opens it again. The fourth student then does the same thing for the lockers numbered with multiple of 4, and so on, down to the hundreth student. In the end, how many lockers are still open?

(a) 1 **(b)** all of the lockers that are not numbered with prime numbers **(c)** 10

(d) 15 **(e)** None of the above

Solution **(c)**: Consider locker number n. Notice that student m will touch the locker only if n is a multiple of m. So we want to consider all factors of n, including 1 and n. The complete list of factors represents all students who will touch the locker.

If n has an even number of factors, then an even number of students will touch the locker. Since the first student opens it, the nth student will close it and it will remain closed. If n has an odd number of factors, then similarly we see that the nth student will open the locker and it will remain open.

It remains to count how many n have an odd number of factors. Each factorization $n = ab$ with $a \neq b$ gives us a pair of factors, keeping the count of factors even. So the only way to get an odd number of factors is if $n = c^2$ for some c, that is if n is a perfect square.

There are 10 perfect squares (12, 22, . . . , 102) between 1 and 100, and so 10 lockers will remain open.

11. Which of the following is the set of all real solutions to the equation $2^{3x} - 5\left(2^{2x}\right) + 2^{x+2} + 6 = 0$?

(a) $\left\{\frac{\ln(3)}{\ln(2)}, \frac{\ln(2-\sqrt{3})}{\ln(2)}\right\}$ **(b)** $\left\{\frac{\ln(3)}{\ln(2)}, \frac{\ln(1+\sqrt{3})}{\ln(2)}\right\}$ **(c)** $\left\{\frac{\ln(1+\sqrt{3})}{\ln(2)}, \frac{\ln(1+\sqrt{5})}{\ln(2)}\right\}$

(d) $\left\{\frac{\ln(3)}{\ln(2)}, \frac{\ln(1-\sqrt{3})}{\ln(2)}, \frac{\ln(1+\sqrt{3})}{\ln(2)}\right\}$ **(e)** None of the above

Solution **(b)**: Substitute $u = 2^x$ to get $u^3 - 5u^2 + 4u + 6 = 0$. Using the Rational Roots Theorem, you can factor the left hand side as $(u-3)(u^2-2u-2) = 0$, and using the quadratic formula, you obtain the solutions $u = 3$ and $u = 1 \pm \sqrt{3}$. Since $u = 2^x$, it has to be positive, which rules out $u = 1 - \sqrt{3}$. The remaining two values for u will give us

$$2^x = 3 \qquad\qquad 2^x = 1 + \sqrt{3}$$
$$x = \frac{\ln 3}{\ln 2} \qquad\qquad x = \frac{\ln\left(1+\sqrt{3}\right)}{\ln 2}$$

12. The line $3x + y = b$ and the equation $2x^2 + y^2 = 1$ are graphed on an (x, y)-rectangular coordinate system. For what values of b will the graph of the line $3x + y = b$ be tangent to the graph of the equation $2x^2 + y^2 = 1$?

(a) The graph of the line $3x + y = b$ is not tangent to the graph of $2x^2 + y^2 = 1$ for any b.

(b) The graph of the line $3x + y = b$ is tangent to the graph of $2x^2 + y^2 = 1$ for all b.

(c) $b = \pm 1$ **(d)** $b = \pm\frac{\sqrt{22}}{2}$ **(e)** None of the above

Solution **(d)**: The line will be tangent to the ellipse if the two curve intersect at exactly one point. So we are looking for the values of the parameter b such that the system of equations

$$\begin{cases} 2x^2 + y^2 & = 1 \\ 3x + y & = b \end{cases}$$

has exactly one solution. Solving the second equation for y and plugging into the first equation gives us

$$2x^2 + (b - 3x)^2 = 1$$
$$11x^2 - 6bx + b^2 - 1 = 0$$

Using the quadratic formula

$$x = \frac{6b \pm \sqrt{36b^2 - 4 \cdot 11 \cdot (b^2 - 1)}}{22} = \frac{6b \pm \sqrt{44 - 8b^2}}{22}$$

To have exactly one solution, $44 - 8b^2$ must be 0. Therefore

$$b^2 = \frac{22}{4}$$
$$b = \pm\frac{\sqrt{22}}{2}$$

13. Consider all three digit numbers for which the tens digit is the sum of the ones digit and the hundreds digit. How many such numbers are divisible by 11? (Note that 0 cannot be the first digit of a three digit number.)

(a) 9 **(b)** 10 **(c)** 45 **(d)** 55 **(e)** None of the above

SOLUTION **(c)**: Let h be the hundreds digit and o the ones digit. Then the tens digit must be $h + o$ and the number will be

$$100h + 10(h + o) + o = 100h + 10h + 10o + o = 110h + 11o = 11(10h + o)$$

therefore *all* such numbers are divisible by 11. We will get one such number for each pair or h and o such that $h + o \leq 9$.

If $h = 1$, there is 9 choices for o: $0, 1, 2, \ldots 8$, if $h = 2$, we only have 8 choices for o etc. All together we have $9 + 8 + 7 + \cdots + 2 + 1 = \frac{9 \cdot 10}{2} = 45$ such numbers.

14. Simplify

$$\left(\frac{1}{\sqrt{a} + \sqrt{a+1}} + \frac{1}{\sqrt{a} - \sqrt{a-1}}\right) \div \left(1 + \sqrt{\frac{a+1}{a-1}}\right) \quad \text{where } a > 1$$

(a) $\sqrt{a} + \sqrt{a+1}$ **(b)** $\frac{1}{\sqrt{a}-\sqrt{a-1}}$ **(c)** $\sqrt{a-1}$ **(d)** $\frac{1}{\sqrt{a+1}}$

(e) None of the above

SOLUTION **(c)**:

$$\begin{aligned}
\frac{\frac{1}{\sqrt{a}+\sqrt{a+1}} + \frac{1}{\sqrt{a}-\sqrt{a-1}}}{1 + \sqrt{\frac{a+1}{a-1}}} &= \frac{\frac{\sqrt{a}-\sqrt{a+1}}{a-(a+1)} + \frac{\sqrt{a}+\sqrt{a-1}}{a-(a-1)}}{\frac{\sqrt{a-1}+\sqrt{a+1}}{\sqrt{a-1}}} \\
&= \frac{\sqrt{a+1} + \sqrt{a-1}}{\frac{\sqrt{a+1}+\sqrt{a-1}}{\sqrt{a-1}}} \\
&= \sqrt{a-1}
\end{aligned}$$

15. Find the sum of all irreducible fractions with a denominator 3 that are between the positive integers m and n, where $m < n$.

(a) $\frac{m(n-m)}{3}$ **(b)** $n^2 - m^2$ **(c)** $\frac{n^2-m^2}{3}$ **(d)** $n^2 - m^2 - 2m$

(e) None of the above

Solution **(b)**: All fractions with denominator 3 that are between the positive integers m and n are:

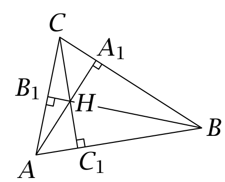
$$\begin{array}{ccc} \frac{3m}{m} & \frac{3m+1}{3} & \frac{3m+2}{3} \\ \frac{3(m+1)}{3} & \frac{3(m+1)+1}{3} & \frac{3(m+1)+2}{3} \\ \vdots & \vdots & \vdots \\ \frac{3(n-1)}{3} & \frac{3(n-1)+1}{3} & \frac{3(n-1)+2}{3} \\ \frac{3n}{3} & & \end{array}$$

The fractions in the first column are reducible. Combining all the fractions in the second and third column we get

$$\frac{2 \cdot 3 \cdot [m + \cdots + (n-1)] + 3 \cdot [n - 1 - m + 1]}{3} = 2\left[\frac{(n-1)n}{2} - \frac{(m-1)}{2}\right] + n - m$$

which after simplifying gives us $n^2 - m^2$.

16. The heights (altitudes) AA_1, BB_1, CC_1 in the triangle ABC intersect at the point H so that $CH = HC_1$. If $\alpha = \angle CAB$ and $\beta = \angle CBA$ then $\tan\alpha\tan\beta$ is



(a) $\frac{1}{2}$ **(b)** 2 **(c)** $\frac{1}{4}$ **(d)** 4 **(e)** None of the above

Solution **(b)**: The triangles BA_1A and HC_1A are similar. Therefore $m\angle ABA_1 = m\angle AHC_1 = \beta$. Then

$$\tan\alpha\tan\beta = \frac{CC_1}{C_1A}\frac{AC_1}{C_1H} = \frac{CC_1}{C_1H} = 2.$$

17. A license plate code consists of two letters followed by three digits. The letters cannot repeat, but the numbers can. What is the probability that a randomly chosen plate has at least one zero?

(a) $\frac{3}{10}$ **(b)** $\frac{1}{10}$ **(c)** $\frac{729}{1000}$ **(d)** $\frac{271}{1000}$ **(e)** None of the above

Solution **(d)**: Since none of the letters can be zero, the letters do not matter. We are looking for the probability of at lest one zero in three digits.

$$P(\text{at least one zero}) = 1 - P(\text{no zero}) = 1 - \frac{9}{10}\frac{9}{10}\frac{9}{10} = \frac{271}{1000}$$

18. How many integers between 1 and 1000 inclusive are neither multiples of 4 nor multiples of 7?

    (a) 357 **(b)** 663 **(c)** 608 **(d)** 643 **(e)** None of the above

    SOLUTION **(d)**: Since $1000/4 = 250$, there are exactly 250 multiples of 4 between 1 and 1000: $1 \cdot 4$, $2 \cdot 4$, $\ldots 250 \cdot 4$. Since $1000/7 \approx 142.9$, there are exactly 142 multiples of 7 between 1 and 1000. However, all the multiples of 28 were on both lists. Since $1000/28 \approx 35.7$, there are exactly 35 such multiples. Therefore there are $250 + 142 - 35 = 357$ numbers between 1 and 1000 that are multiple of 4 or multiples of 7. Therefore there are $1000 - 357 = 643$ numbers between 1 and 1000 that are neither multiples of 4 nor multiples of 7.

19. Suppose that $f(x) = 2(f(x+1) + f(x-1))$ for all x. If $f(2) = 2$ and $f(4) = -2$, what is $f(7)$?

    (a) -1 **(b)** $\frac{3}{2}$ **(c)** $\frac{7}{4}$ **(d)** $-\frac{5}{8}$ **(e)** None of the above

    SOLUTION **(c)**: First, $f(3) = 2(f(4)+f(2)) = 2(2-2) = 0$. Solving the relation $f(x) = 2(f(x+1) + f(x-1))$ for $f(x+1)$ gives us

    $$f(x+1) = \frac{1}{2}f(x) - f(x-1)$$

    Therefore

    $$\begin{aligned} f(5) &= \frac{1}{2}f(4) - f(3) = -1 - 0 = -1 \\ f(6) &= \frac{1}{2}f(5) - f(4) = -\frac{1}{2} + 2 = \frac{3}{2} \\ f(7) &= \frac{1}{2}f(6) - f(5) = \frac{3}{4} + 1 = \frac{7}{4} \end{aligned}$$

20. Suppose $p(x)$ is a degree 4 polynomial with rational coefficients. If 1 is the only real root of $p(x)$, but $p(x)$ is not a multiple of $(x-1)^4$, which of the following must also be true?

    (a) 1 must be a root of multiplicity one.

    (b) $p(x)$ has two roots of multiplicity two.

    (c) 1 is the only root of $p(x)$ of multiplicity two.

    (d) 1 must be a root of multiplicity three.

    (e) Not enough information is given.

    SOLUTION **(c)**: Any polynomial of degree 4 must have exactly 4 roots, when counted with multiplicity. Since $p(x)$ is not a multiple of $(x-1)^4$, 1 is not the only root of the polynomial. It is the only real root, though, so all other roots must be non-real complex number. Since $p(x)$ has only real coefficient, its non-real roots come in complex conjugate pairs. Therefore there must be an even number of them, in this case two, each of them of multiplicity one. Therefore 1 must have multiplicity 2.

21. Find the value of $\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right) + \tan\left(\cos^{-1}\left(\frac{5}{7}\right)\right)$.

    (a) $\frac{11}{5}$ **(b)** $\frac{4+2\sqrt{6}}{5}$ **(c)** $\frac{\pi}{6}$ **(d)** $\frac{8\sqrt{6}+25}{10\sqrt{6}}$ **(e)** None of the above

    SOLUTION **(b)**:

22. The equation $\log x^2 = (\log x)^2$ has

(a) no solution **(b)** one solution **(c)** two solutions

(d) three solutions **(e)** more than three solutions

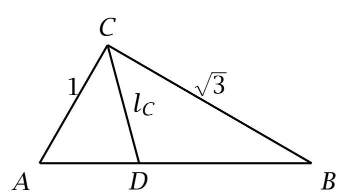
SOLUTION **(c)**: Using properties of logarithms, the equation can be rewritten as $2\log x = (\log x)^2$. Substituting $u = \log x$ gives us $2u = u^2$ or $2u - u^2 = 0$ or $u(2-u) = 0$, which means that $u = 0$ or $u = 2$. That gives us $x = 1$ or $x = 200$.

23. If $a^y = x$, what is $\log_x \frac{1}{a}$? (Assume $a > 0$, $x > 0$ and $x \neq 1$.)

(a) $\frac{1}{y}$ **(b)** $-y$ **(c)** $-a$ **(d)** $-\frac{1}{y}$ **(e)** None of the above

SOLUTION **(d)**:

24. A triangle ABC has sides $\overline{AB}$ of length 2 ft, $\overline{AC}$ of length 1 ft and $\overline{BC}$ of length $\sqrt{3}$ ft. The angle bisector l_C of $\angle ACB$ intersects the side AB at the point D. Find AD.



(a) $\sqrt{3} - 1$

(b) $\sqrt{3} + 1$ **(c)** $\frac{2}{3}$ **(d)** $\frac{2}{\sqrt{3}}$ **(e)** None of the above

SOLUTION **(a)**:

25. A couple plans to have 6 children. Assuming that boys and girls are equally likely to be born, what is the probability that they will have exactly 3 of each?

(a) $\frac{5}{16}$ **(b)** .5 **(c)** $\frac{1}{32}$ **(d)** $\frac{1}{8}$ **(e)** None of the above

SOLUTION **(a)**: