2008 Math Olympics – Level II Solutions

1. Find all solutions to the equation: $\log_3(x-1)+\log_3(x+7)=2$

(a) -8 and 2 **(b)** 2 **(c)** -2 **(d)** 1 **(e)** None of the above

SOLUTION **(b)**: The equation is equivalent to

$$\log_3[(x-1)(x+7)]=2,\ x-1>0 \text{ and } x+7>0.$$

Further manipulation gives us

$$\begin{aligned}(x-1)(x+7)&=9, x>1\\ x^2+6x-16&=0, x>1\\ (x+8)(x-2)&=0, x>1\\ x&=2\end{aligned}$$

2. Compute $\dfrac{\sin 80^\circ}{\sin 20^\circ+\sin 40^\circ}$

(a) 1 **(b)** 0 **(c)** $\frac{\sqrt{2}}{2}$ **(d)** $\sin 20^\circ$ **(e)** None of the above

SOLUTION **(a)**:

$$\begin{aligned}\frac{\sin 80^\circ}{\sin 20^\circ+\sin 40^\circ}&=\frac{2\sin 40^\circ\cos 40^\circ}{\sin 20^\circ+2\sin 20^\circ\cos 20^\circ}=\frac{2\left(2\cancel{\sin 20^\circ}\cos 20^\circ\right)\cos 40^\circ}{\cancel{\sin 20^\circ}\left(1+2\cos 20^\circ\right)}\\ &=\frac{4\cos 20^\circ\cos 40^\circ}{2\left(\frac{1}{2}+\cos 20^\circ\right)}=\frac{4\cos 20^\circ\cos 40^\circ}{2\left(\cos 60^\circ+\cos 20^\circ\right)}\\ &=\frac{\cancel{4}\cos 20^\circ\cos 40^\circ}{\cancel{2\cdot 2}\cos\frac{60^\circ+20^\circ}{2}\cos\frac{60^\circ-20^\circ}{2}}=\frac{\cos 20^\circ\cos 40^\circ}{\cos 40^\circ\cos 20^\circ}=1\end{aligned}$$

3. The strange operation $*$ is defined to be:

$$a * b := \frac{a}{a + \frac{1}{b}}$$

where a and b are real numbers. Which of the following is true?

(a) $*$ is associative but not commutative.

(b) $*$ is commutative but not associative.

(c) $*$ is both associative and commutative.

(d) $*$ is neither associative nor commutative.

(e) $*$ is not associative and is commutative only if $a = 1$ or $b = 1$.

Solution **(b)**: Simplifying the complex fraction, $a * b = \frac{ab}{ab+1} = \frac{ba}{ba+1} = b * a$. Then

$$(a * b) * c = \frac{ab}{ab + 1} * c = \frac{\frac{abc}{ab+1}}{\frac{abc}{ab+1} + 1} = \frac{abc}{abc + ab + 1}$$

while

$$a * (b * c) = a * \frac{bc}{bc + 1} = \frac{\frac{abc}{bc+1}}{\frac{abc}{bc+1} + 1} = \frac{abc}{abc + bc + 1}.$$

Therefore if $a \neq c$, $(a * b) * c \neq a * (b * c)$.

4. Let x_1 and x_2 be the roots of the quadratic equation $x^2 + px + q = 0$. If

$$x_1 = \frac{x_2 + 2}{2x_2 - 1},$$

which of the following expresses the relation between p and q?

(a) $2q + p = 2$ **(b)** $2q - p = 2$

(c) $-2q + p = 2$ **(d)** the relation cannot be determined

(e) None of the above

Solution **(a)**: If x_1 and x_2 are solutions to the equation $x^2 + px + q = 0$ then $x_1 + x_2 = -p$ and $x_1x_2 = q$. Since

$$x_1 = \frac{x_2 + 2}{2x_2 - 1},$$

we have that

$$\begin{aligned}
\frac{x_2 + 2}{2x_2 - 1} + x_2 &= -p \\
\frac{x_2 + 2}{2x_2 - 1} + x_2 + p &= 0 \\
x_2 + 2 + 2x_2^2 - x_2 + 2px_2 - p &= 0 \\
x_2^2 + px_2 + 1 - \frac{p}{2} &= 0 \\
x_2^2 + px_2 &= \frac{p}{2} - 1
\end{aligned}$$

Since x_2 is a solution of the equation $x^2 + px + q = 0$, we know that $x_2^2 + px_2 = -q$, and so $q = 1 - \frac{p}{2}$, or $2q + p = 2$.

Note that you can obtain the same result by using $x_1x_2 = q$ instead of $x_1 + x_2 = -p$.

5. Simplify the following expression: $2\log_a(\cos\theta) + 2\log_a(\sec\theta)$

(a) $\log_a \tan^2\theta$ **(b)** $\log_a 2$ **(c)** 1 **(d)** 0

(e) None of the above

Solution **(d)**:

$$\begin{aligned}
2\log_a(\cos\theta) + 2\log_a(\sec\theta) &= 2\log_a(\cos\theta) + 2\log_a\left(\frac{1}{\cos\theta}\right) \\
&= 2\log_a(\cos\theta) + 2\log_a(1) - 2\log_a(\cos\theta) \\
&= 2\log_a(1) = 0
\end{aligned}$$

6. Find all solutions to the equation: $\cos(\ln x) = 0$

(a) $e^{\frac{\pi}{2}+n\pi}$ **(b)** $2n\pi$ **(c)** $\frac{\pi}{2}+n\pi$ **(d)** $e^{2n\pi}$ **(e)** None of the above

SOLUTION **(a)**:

$$\begin{aligned} \cos(\ln x) &= 0 \\ \ln x &= \frac{(2n+1)\pi}{2} \text{ for } n \in \mathbb{Z} \\ x &= e^{\frac{(2n+1)\pi}{2}} = e^{n\pi+\frac{\pi}{2}} \text{for } n \in \mathbb{Z}. \end{aligned}$$

7. Simplify the following expression: $\log_a b \cdot \log_b c \cdot \log_c d \cdot \log_d a$

(a) $\dfrac{\log_a b}{\log_c d}$ **(b)** $\log_a d$ **(c)** 1 **(d)** 0 **(e)** None of the above

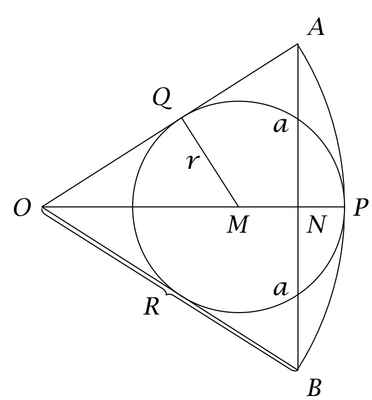
SOLUTION **(c)**: According to the change of base formula,

$$\log_b x = \frac{\log_a x}{\log_a b}$$

which means that $\log_a b \cdot \log_b x = \log_a x$. Applying this formula repeatedly gives us

$$\log_a b \cdot \log_b c \cdot \log_c d \cdot \log_d a = \log_a c \cdot \log_c d \cdot \log_d a = \log_a d \cdot \log_d a = \log_a a = 1$$

8. Which of the following express the circumference of a circle inscribed in a sector OAB with radius R and $AB = 2a$?



(a) $2\pi\dfrac{Ra}{R+a}$

(b) $\dfrac{2\pi R^2}{9}$ **(c)** $2\pi(R-a)^2$

(d) $2\pi\dfrac{R}{R-a}$ **(e)** None of the above

SOLUTION **(a)**: The triangles $\triangle OQM$ and $\triangle ONA$ are similar, which means that $\frac{QM}{AN} = \frac{OM}{OA}$, or $\frac{r}{a} = \frac{OM}{R} = \frac{R-r}{R} = 1-\frac{r}{R}$, or $\frac{r}{a}+\frac{r}{R} = \frac{r(R+a)}{Ra} = 1$, or

$$r = \frac{Ra}{R+a}.$$

Multiplying by 2π will give us the circumference.

9. Mary and John wrote 100 numbers each. Mary's sequence starts with $5, 8, 11, 14, \ldots$, John's sequence starts with $3, 7, 11, 15, \ldots$. How many common numbers are there in both sequences?

(a) 20 **(b)** 21 **(c)** 25 **(d)** 30 **(e)** None of the above

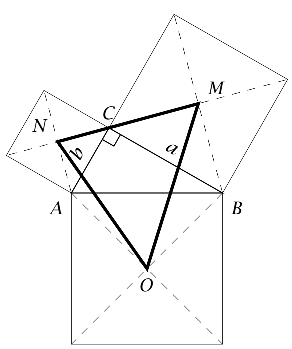
Solution **(c)**: Mary's sequence is $5 + 3k_1$, $0 \le k_1 \le 99$, John's sequence is $3 + 4k_2$, $0 \le k_2 \le 99$. We need to find when $5 + 3k_1 = 3 + 4k_2$ for $0 \le k_1, k_2 \le 99$.

$$\begin{aligned} 5 + 3k_1 &= 3 + 4k_2 \\ 2 + 3k_1 &= 4k_2 \\ \frac{2 + 3k_1}{4} &= k_2 \\ k_1 + \frac{2 - k_1}{4} &= k_2 \\ \frac{2 - k_1}{4} &= k_2 - k_1 \end{aligned}$$

which means that $t = \frac{2-k_1}{4}$ must be an integer. Then $k_1 = -4t + 2$ and $k_2 = k_1 + t = -3t + 2$. Since $0 \le k_1, k_2 \le 99$, we have

$$\begin{aligned} 0 \le -4t + 2 &\le 99 \\ -2 \le -4t &\le 97 \\ \frac{1}{2} \ge t &\ge -24\frac{1}{4} \end{aligned} \qquad\qquad \begin{aligned} 0 \le -3t + 2 &\le 99 \\ -2 \le -3t &\le 97 \\ \frac{1}{3} \ge t &\ge -32\frac{1}{3} \end{aligned}$$

There ate 25 integers t that satisfy both of the compound inequalities: $t = -24, -23, \ldots, -1, 0$. Therefore there are 25 common numbers in both sequences.

10. 

The legs of a right triangle $\triangle ABC$ are a and b ($\angle C = 90°$, $BC = a$ and $AC = b$). The points M, N and O are the centers of the squares of $\triangle ABC$ with sides BC, AC and AB, respectively. Find the area of $\triangle MNO$.

(a) $\frac{1}{4}(a + b)^2$

(b) $\dfrac{a^2 + b^2}{4}$ **(c)** $2\sqrt{a^2 + b^2}\sqrt{a^2 - b^2}$

(d) $\dfrac{a^2 + b^2}{2}$ **(e)** None of the above

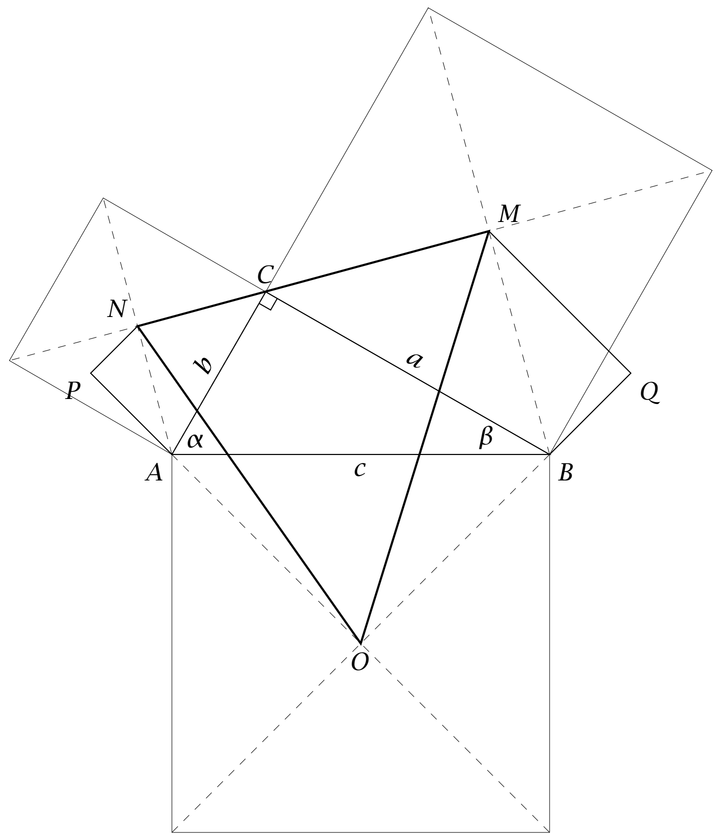
Solution **(a)**: First notice that the point C indeed does lie on the segment MN, while the points A and B will always be outside of the triangle $\triangle MNO$. Then

$$A_{\triangle MNO} = A_{AOBMN} - A_{\triangle AON} - A_{\triangle BOM}.$$

Furthermore

$$A_{AOBMN} = A_{\triangle ABC} + A_{\triangle ABO} + A_{\triangle BCM} + A_{\triangle ACN} = \frac{1}{2}ab + \frac{1}{4}\left(\sqrt{a^2 + b^2}\right)^2 + \frac{1}{4}a^2 + \frac{1}{4}b^2.$$

Let $\alpha = \angle CAB$, $\beta = \angle CBA$ and $c = AB$. Add points P and Q such that PN is the altitude of $\triangle AON$ perpendicular to AO and QM is the altitude of $\triangle OBM$ perpendicular to OB, as shown in the picture:



Then $\angle PAN = 180^\circ - (45^\circ + \alpha + 45^\circ) = 90^\circ - \alpha = \beta$ and $\angle QBM = 180^\circ - (45^\circ + \beta + 45^\circ) = 90^\circ - \beta = \alpha$. Therefore $\triangle NAP$ and $\triangle BMQ$ are both similar to $\triangle ABC$, and

$$\frac{PN}{AN} = \frac{b}{c}, \text{ and } \frac{QM}{BM} = \frac{a}{c}.$$

Then

$$A_{\triangle AON} = \frac{1}{2}AO \cdot PN = \frac{1}{2}AO \cdot \frac{b}{c}AN = \frac{1}{2}\frac{\sqrt{2}}{2}c \cdot \frac{b}{c}\frac{\sqrt{2}}{2}b = \frac{1}{4}b^2$$

and

$$A_{\triangle BOM} = \frac{1}{2}BO \cdot QM = \frac{1}{2}BO \cdot \frac{a}{c}BM = \frac{1}{2}\frac{\sqrt{2}}{2}c \cdot \frac{a}{c}\frac{\sqrt{2}}{2}a = \frac{1}{4}a^2$$

Therefore

$$A_{\triangle MNO} = \frac{1}{2}ab + \frac{1}{4}\left(\sqrt{a^2 + b^2}\right)^2 = \frac{1}{4}(a + b)^2.$$

11. Find the number of solutions of the system

$$\begin{cases} 3x^2 - 2xy + y^2 = 36 \\ 5x^2 - 4xy + y^2 = 20 \end{cases}$$

(a) None **(b)** One **(c)** Two **(d)** Four **(e)** Five

SOLUTION **(d)**: First note that there is no solution of the system with $y = 0$.

$$\begin{cases} 3x^2 - 2xy + y^2 = 36 \\ 5x^2 - 4xy + y^2 = 20 \end{cases}$$

$$\begin{cases} 15x^2 - 10xy + 5y^2 = 180 \longleftarrow \text{multiply by 5} \\ 45x^2 - 36xy + 9^2 = 180 \longleftarrow \text{multiply by 9} \end{cases}$$

Subtract the first equation from the second one:

$$30x^2 + 26xy - 4y^2 = 0$$

$$15x^2 - 13xy + 2y^2 = 0 \qquad \div 2$$

$$15\left(\frac{x}{y}\right)^2 - 13\left(\frac{x}{y}\right) + 2 = 0 \qquad \div y \neq 0$$

Solve the quadratic equation for $\frac{x}{y}$:

$$\frac{x}{y} = \frac{13 \pm \sqrt{13^2 - 4 \cdot 15 \cdot 2}}{2 \cdot 15} = \frac{13 \pm 7}{30} = \begin{cases} \frac{2}{3} & \text{i.e. } \frac{x}{y} = \frac{2}{3} \\ \frac{1}{5} & \text{i.e. } \frac{x}{y} = \frac{1}{5} \end{cases}$$

(a) If $x = \frac{2}{3}y$, then from the first equation we get

$$3\left(\frac{4}{9}y^2\right) - 2\left(\frac{2}{3}y\right)y + y^2 = 36$$

or $y^2 = 36$ or $y = \pm 6$. The two solutions corresponding to this case are $(4, 6)$ and $(-4, -6)$.

(b) If $y = 5x$ then

$$3x^2 - 2x(5x) + 25x^2 = 36$$

or $18x^2 = 36$ or $x = \pm\sqrt{2}$. The two solutions in this case are $(\sqrt{2}, 5\sqrt{2})$ and $(-\sqrt{2}, -5\sqrt{2})$.

12. How many *integer* solutions does the equation $(x+2)(x-4)(x+6)(x-8) = 225$ have?

(a) None **(b)** One **(c)** Two **(d)** Three **(e)** Four

SOLUTION **(c)**:

$$\begin{aligned}
(x+2)(x-4)(x+6)(x-8) &= 225 && \text{Partially expand}\\
(x^2-2x-8)(x^2-2x-48) &= 225 && \text{Subst. } y = x^2-2x\\
(y-8)(y-48) &= 225 && \text{Expand}\\
y^2-56y+384 &= 225 && -225\\
y^2-56y+159 &= 0 && \text{Factor}\\
(y-3)(y-53) &= 0
\end{aligned}$$

Split into two equations and substitute back:

$$\begin{aligned}
y-3 &= 0 \\
x^2-2x-3 &= 0 \\
(x+1)(x-3) &= 0
\end{aligned}
\qquad\qquad
\begin{aligned}
y-53 &= 0 \\
x^2-2x-53 &= 0
\end{aligned}$$

The first equation will have solutions $x = -1$ and $x = 3$ and the second equation will have solutios

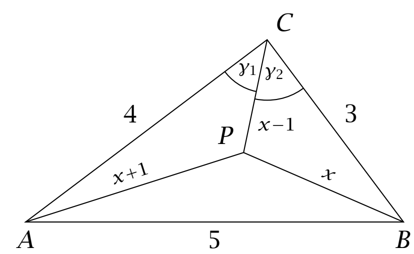
$$x = \frac{2 \pm \sqrt{4-4(-53)}}{2} = 1 \pm 3\sqrt{6}$$

Only two of the solutions are integers.

13. The legs AC and BC of a right triangle $\triangle ABC$ ($\angle C = 90°$) are 4 and 3 units, respectively. The point P is chosen inside $\triangle ABC$ so that $AP + BC = BP + CA = CP + AB$. Find the distance from P to B.

(a) $\frac{52}{23}$ **(b)** $\frac{75}{23}$ **(c)** $\frac{29}{23}$ **(d)** 2 **(e)** None of the above

SOLUTION **(a)**:
$AB = \sqrt{4^2 + 3^2} = 5$. Let $x = PB$. Then since $AP + BC = BP + CA = CP + AB$ we have that $PA = x + 1$ and $PC = x - 1$. According to the law of cosines for the triangle $\triangle ACP$ and $\triangle BCP$,



$$(x+1)^2 = 4^4 + (x-1)^2 - 8(x-1)\cos\gamma_1$$
$$x^2 = (x-1)^2 + 3^3 - 6(x-1)\cos\gamma_2$$

from which we obtain

$$\cos\gamma_1 = \frac{4-x}{2(x-1)}$$
$$\cos\gamma_2 = \frac{5-x}{3(x-1)}$$

But $\gamma_1 + \gamma_2 = 90°$, so $\cos\gamma_2 = \sin\gamma_1$ and $\cos^2\gamma_1 + \cos^2\gamma_2 = \cos^2\gamma_1 + \sin^2\gamma_1 = 1$, or

$$\left(\frac{4-x}{2(x-1)}\right)^2 + \left(\frac{5-x}{3(x-1)}\right)^2 = 1.$$

This reduces to $23x^2 + 40x - 208 = 0$, and solving this will give us $x = \frac{52}{23}$ and $x = -4$.

14. Each edge of a cube is increased by 50%. What is the percent increase in the surface area of the cube?

(a) 50% **(b)** 125% **(c)** 150% **(d)** 300% **(e)** None of the above

SOLUTION **(b)**: Suppose the side of the cube was originally a. Then the surface area was $6a^2$. The side of the increased cube is $1.5a$, so the surface area of the increases cube is

$$6(1.5a)^2 = 6 \cdot 2.25 \cdot a^2 = 2.25(6a^2)$$

so the surface area increased by 125%.

15. If $f(x) = x^3 + 1$ and c is a positive number such that $f(c+2) = f(c) + f(2)$ then c equals:

(a) $\sqrt{\frac{30}{3}} + 1$ **(b)** $\sqrt{\frac{7}{6}} + 1$ **(c)** $\sqrt{10} - 1$ **(d)** $\sqrt{\frac{7}{6}} - 1$ **(e)** None of the above

SOLUTION **(d)**:

$$\begin{aligned} f(c+2) &= f(c) + f(2) \\ (c+2)^3 + 1 &= c^3 + 1 + 2^3 + 1 \\ c^3 + 6c^2 + 12c + 9 &= c^3 + 10 \\ 6c^2 + 12c - 1 &= 0 \end{aligned}$$

so

$$c = \frac{-12 \pm \sqrt{12^2 - 4 \cdot 6 \cdot (-1)}}{2 \cdot 6} = \frac{-12 \pm 2\sqrt{42}}{12} = -1 \pm \sqrt{\frac{42}{36}} = -1 \pm \sqrt{\frac{7}{6}}$$

Since $\sqrt{\frac{7}{6}} > 1$, the positive solution is $\sqrt{\frac{7}{6}} - 1$.

16. $\log_2 (\log_3 (\log_4 2^n)) = 2$. Find n.

(a) $\frac{27}{\log_4 2}$ **(b)** $\frac{4^{81}}{2}$ **(c)** 162 **(d)** 24 **(e)** None of the above

SOLUTION **(c)**:

$$\begin{aligned} \log_3 (\log_4 2^n) &= 4 \\ \log_4 2^n &= 81 \\ 4^{81} &= 2^n \\ 2^{2 \cdot 81} &= 2^n \\ n &= 162 \end{aligned}$$

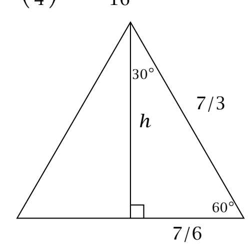
17. A square and an equilateral triangle both have perimeter of 7 cm. How much bigger is the area of the square than the area of the triangle?

(a) $\frac{49}{2}\left(\frac{1}{8} - \frac{\sqrt{3}}{9}\right)$ cm^2 **(b)** $\frac{49}{16} - \frac{49}{36}$ cm^2 **(c)** $\frac{49}{4}\left(\frac{1}{4} - \frac{\sqrt{3}}{9}\right)$ cm^2

(d) $\frac{49}{16} - \frac{49}{18}$ cm^2 **(e)** None of the above

SOLUTION **(c)**: Let a be the length of side of the square. Then $a = \frac{7}{4}$ and $A_{\text{square}} = \left(\frac{7}{4}\right)^2 = \frac{49}{16}$.

Let b be the length of side of the triangle. Then $b = \frac{7}{3}$, and the height of the triangle is $h = \frac{7\sqrt{3}}{6}$ (see the picture). So



$$A_{\text{triangle}} = \frac{1}{2} \cdot \frac{7}{3} \cdot \frac{7\sqrt{3}}{6} = \frac{49\sqrt{3}}{36}$$

and

$$A_{\text{square}} - A_{\text{triangle}} = \frac{49}{16} - \frac{49\sqrt{3}}{36} = \frac{49}{4}\left(\frac{1}{4} - \frac{\sqrt{3}}{9}\right)$$

18. The operations of addition and multiplications of sets and sets and sets and numbers are defined in the following way: if A and B are sets of numbers and c is a number,

$$\begin{aligned} A + B &= \{x + y | x \in A \text{ and } y \in B\} \\ A \cdot B &= \{x \cdot y | x \in A \text{ and } y \in B\} \\ A + c &= \{x + c | x \in A\} \\ c \cdot A &= \{c \cdot x | x \in A\} \end{aligned}$$

Let $E = \{0, \pm 2, \pm 4, \dots\}$ be the set of all even integers. Which of the following statements is false?

(a) $E + 1 = E + (-1)$ **(b)** $E + E = 2 \cdot E$

(c) $E = -1 \cdot E$ **(d)** More than one of them are false

(e) None of the statements is false

Solution **(b)**:

$$E + 1 = \{x + 1 | x \in E\} = \{x + 1 | x \text{ is an even integer}\}$$

which is the set of all odd integers.

$$E + (-1) = \{x - 1 | x \in E\} = \{x - 1 | x \text{ is an even integer}\}$$

which is also the set of all odd integers. So the first statement is true.

$$E + E = \{x + y | x \in E \text{ and } y \in E\} = \{x + y | x \text{ and } y \text{ are both even}\}$$

Clearly every element of $E + E$ is even, as it is a sum of two even numbers. On the other hand, every even number x can be written as $x + 0$, and therefore is in $E + E$. Therefore $E + E = E$. On the other hand

$$2 \cdot E = \{2x | x \in E\} = \{2x | x \text{ even}\} = \{4k | k \in \mathbb{Z}\}$$

which is a set of all multiples of 4. Therefore the second statement is false.

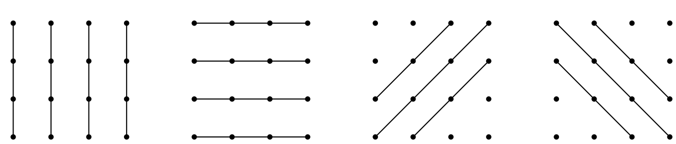
$$-1 \cdot E = \{-x | x \in E\}$$

which is again the set of all even numbers, so the third statement is true.

19. How many triangles with positive area are there whose vertices are points in the xy-plane with integer coordinates (x, y) satisfying $1 \le x \le 4$ and $1 \le y \le 4$?

(a) 496 **(b)** 500 **(c)** 512 **(d)** 516 **(e)** 560

Solution **(d)**: There are 16 points with integer coordinates satisfying given conditions. Each set of three such points will be vertices for a triangle with positive area if and only if the three points are not collinear. Altogether there are $\binom{16}{3} = 560$ sets of three points. Out of those the ones that are collinear lie either on the vertical liones, the horizontal lines, or on lines with slope 1 or -1:



There are $\binom{4}{3}$ sets of three points on each of the vertical or horizontal lines. There are $1 + \binom{4}{3} + 1$ sets on the lines with slope 1 and $1 + \binom{4}{3} + 1$ sets on the lines with slope -1. Altogether there are $10\binom{4}{3} + 4 = 44$ collinear sets. That leaves us with $560 - 44 = 516$ triangles with non-zero area.

20. Two copies of the parabola with equation $y = x^2$ are modified in the following ways:

- The first copy is scaled in the vertical direction by a factor of 2.
- The second copy is scaled in the horizontal direction by a positive factor k.

The resulting graphs are exactly the same. Find k.

(a) $k = 2$ **(b)** $k = \sqrt{2}$ **(c)** $k = \frac{1}{2}$ **(d)** $k = \frac{\sqrt{2}}{2}$ **(e)** None of the above

Solution **(d)**: The equation of parabola scaled in vertical direction by the factor of 2 is $y = 2x^2$. The equation of parabola scaled by the factor k in horizontal direction is $y = \left(\frac{x}{k}\right)^2$. Since the graphs are the same,

$$2x^2 = \left(\frac{x}{k}\right)^2 = \frac{x^2}{k^2}$$

or

$$\frac{1}{k^2} = 2$$

or

$$k = \frac{1}{\sqrt{2}}$$

21. Suppose n and b are positive numbers. If $\log_b n = 2$ and $\log_n 2b = 2$, what is b?

(a) 1 **(b)** $\sqrt{2}$ **(c)** $\sqrt[3]{4}$ **(d)** $\sqrt[3]{2}$ **(e)** None of the above

Solution **(d)**: Rewriting both equations in exponential form gives us $b^2 = n$ and $n^2 = 2b$, so $b^4 = 2b$, or $b^4 - 2b = 0$, or $b(b^3 - 2) = 0$. Since b has to be positive, $b = \sqrt[3]{2}$.

22. Which of the following is not equal to $\sec^2\theta\csc^2\theta$?

(a) $\sec^2\theta + \csc^2\theta$

(b) $\dfrac{\tan^2\theta + 1}{1 - \cos^2\theta}$

(c) $(\tan\theta + \cot\theta)^2$

(d) $\dfrac{1}{\sec^2\theta} + \dfrac{1}{\csc^2\theta}$

(e) More than one of them are not equal to $\sec^2\theta\csc^2\theta$

Solution **(d)**:

- $\sec^2\theta + \csc^2\theta = \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta\sin^2\theta} = \frac{1}{\cos^2\theta}\cdot\frac{1}{\sin^2\theta} = \sec^2\theta + \csc^2\theta$
- $\frac{\tan^2\theta+1}{1-\cos^2\theta} = \frac{\sec^2\theta}{\sin^2\theta} = \sec^2\theta\cdot\frac{1}{\sin^2\theta} = \sec^2\theta\csc^2\theta$
- $(\tan\theta + \cot\theta)^2 = \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right)^2 = (\sec\theta\csc\theta)^2 = \sec^2\theta\csc^2\theta$

- $\frac{1}{\sec^2\theta} + \frac{1}{\csc^2\theta} = \cos^2\theta + \sin^2\theta = 1$, so this is the only one that is not equal to the given expression.

23. What is $(1+i)^{13}$?

(a) $-64-64i$ **(b)** $-8192i$ **(c)** $64\sqrt{2}+64\sqrt{2}i$ **(d)** $-12-12i$

(e) None of the above

Solution **(a)**: Writing the number $z = 1+i$ as $z = \sqrt{2}e^{\frac{\pi}{4}i}$,

$$z^{13} = \left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^{13} = \sqrt{2^{13}}e^{\frac{13\pi}{4}i} = 64\sqrt{2}e^{3\pi i}e^{\frac{\pi}{4}i} = -64\sqrt{2}e^{\frac{\pi}{4}i} = -64z$$

which is $-94(1+i) = -64-64i$.

24. Juniors and Seniors from four schools are going to a State competition. There are 40 students from Atherton High School, 25 from Bayside High, 20 from Clairemont High School and 15 from Devonshire High School. 16 of the students from Atherton, 5 of the students from Bayside, 15 of the students from Clairemont, and 9 from Devonshire, are Juniors. If a Senior is selected at random, what is the probablilty that he or she is from Clairemont?

(a) .05 **(b)** .25 **(c)** $\frac{1}{11}$ **(d)** $\frac{1}{3}$ **(e)** None of the above

Solution **(c)**: We start by filling it the following table:

School	Total	Juniors	Seniors
Atherton	40	16	24
Bayside	25	5	20
Clairemont	20	15	5
Devinshire	15	9	6
Total			55

There are 55 seniors total, 5 of them from Clairemont, so the probability that a randomly selected senior is from Clairemont is $\frac{5}{55} = \frac{1}{11}$.

25. Suppose $f(x) = f\left(\frac{1}{x}\right)$, for all $x \neq 0$, and that $f(a) + f(b) - f(a+b) = \frac{2(a^3-b^3)}{ab(a^2-b^2)}$ for all a and b such that $a^2 \neq b^2$. Suppose also that $f(2) = 5$. What is $f(2.5)$?

(a) 4.2 **(b)** 5.8 **(c)** 7.9 **(d)** 1 **(e)** None of the above

Solution **(b)**: $f(a+b) = f(a) + f(b) - \frac{2(a^3-b^3)}{ab(a^2-b^2)} = f(a) + f(b) - \frac{2(a^2+ab+b^2)}{ab(a+b)}$ Now

$$\begin{aligned} f(2.5) = f(2+.5) &= f(2) + f(.5) - \frac{2\left(2^2+2\cdot .5+.5^2\right)}{2\cdot .5\,(2+.5)} \\ &= 5+5-\frac{2\,(5.25)}{1\,(2.5)} \\ &= 10 - \frac{10.5}{2.5} = 10-4.2 = 5.8 \end{aligned}$$