Saginaw Valley State University

2008 Math Olympics – Level II

1. Find all solutions to the equation: $\log_3(x-1) + \log_3(x+7) = 2$

   (a) -8 and 2 **(b)** 2 **(c)** -2 **(d)** 1 **(e)** None of the above

2. Compute $\dfrac{\sin 80^\circ}{\sin 20^\circ + \sin 40^\circ}$

   (a) 1 **(b)** 0 **(c)** $\dfrac{\sqrt{2}}{2}$ **(d)** $\sin 20^\circ$ **(e)** None of the above

3. The strange operation $*$ is defined to be:

$$a * b := \frac{a}{a + \frac{1}{b}}$$

   where a and b are real numbers. Which of the following is true?

   (a) $*$ is associative but not commutative.

   (b) $*$ is commutative but not associative.

   (c) $*$ is both associative and commutative.

   (d) $*$ is neither associative nor commutative.

   (e) $*$ is not associative and is commutative only if $a = 1$ or $b = 1$.

4. Let x_1 and x_2 be the roots of the quadratic equation $x^2 + px + q = 0$. If

$$x_1 = \frac{x_2 + 2}{2x_2 - 1},$$

   which of the following expresses the relation between p and q?

   (a) $2q + p = 2$ **(b)** $2q - p = 2$

   (c) $-2q + p = 2$ **(d)** the relation cannot be determined

   (e) None of the above

5. Simplify the following expression: $2\log_a(\cos\theta) + 2\log_a(\sec\theta)$

   (a) $\log_a \tan^2\theta$ **(b)** $\log_a 2$ **(c)** 1 **(d)** 0

   (e) None of the above

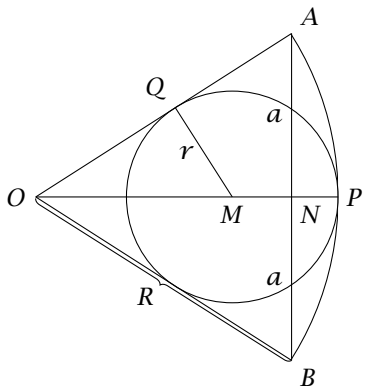
6. Find all solutions to the equation: $\cos(\ln x) = 0$

   (a) $e^{\frac{\pi}{2}+n\pi}$ **(b)** $2n\pi$ **(c)** $\frac{\pi}{2}+n\pi$ **(d)** $e^{2n\pi}$ **(e)** None of the above

7. Simplify the following expression: $\log_a b \cdot \log_b c \cdot \log_c d \cdot \log_d a$

   (a) $\dfrac{\log_a b}{\log_c d}$ **(b)** $\log_a d$ **(c)** 1 **(d)** 0 **(e)** None of the above

8. Which of the following express the circumference of a circle inscribed in a sector OAB with radius R and $AB = 2a$?



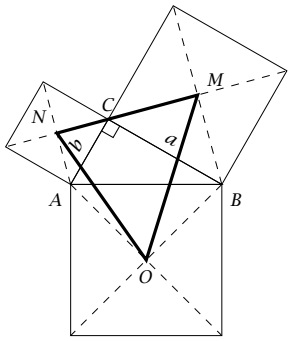
   (a) $2\pi\dfrac{Ra}{R+a}$

   (b) $\dfrac{2\pi R^2}{9}$ **(c)** $2\pi(R-a)^2$

   (d) $2\pi\dfrac{R}{R-a}$ **(e)** None of the above

9. Mary and John wrote 100 numbers each. Mary's sequence starts with $5, 8, 11, 14, \ldots$, John's sequence starts with $3, 7, 11, 15, \ldots$. How many common numbers are there in both sequences?

   (a) 20 **(b)** 21 **(c)** 25 **(d)** 30 **(e)** None of the above

10. 

The legs of a right triangle $\triangle ABC$ are a and b ($\angle C = 90°$, $BC = a$ and $AC = b$). The points M, N and O are the centers of the squares of $\triangle ABC$ with sides BC, AC and AB, respectively. Find the area of $\triangle MNO$.

   (a) $\frac{1}{4}(a+b)^2$

   (b) $\dfrac{a^2+b^2}{4}$ **(c)** $2\sqrt{a^2+b^2}\sqrt{a^2-b^2}$

   (d) $\dfrac{a^2+b^2}{2}$ **(e)** None of the above

11. Find the number of solutions of the system

$$\begin{cases} 3x^2 - 2xy + y^2 = 36 \\ 5x^2 - 4xy + y^2 = 20 \end{cases}$$

   (a) None **(b)** One **(c)** Two **(d)** Four **(e)** Five

12. How many *integer* solutions does the equation $(x+2)(x-4)(x+6)(x-8) = 225$ have?

   (a) None **(b)** One **(c)** Two **(d)** Three **(e)** Four

13. The legs AC and BC of a right triangle $\triangle ABC$ ($\angle C = 90°$) are 4 and 3 units, respectively. The point P is chosen inside $\triangle ABC$ so that $AP + BC = BP + CA = CP + AB$. Find the distance from P to B.

(a) $\frac{52}{23}$ **(b)** $\frac{75}{23}$ **(c)** $\frac{29}{23}$ **(d)** 2 **(e)** None of the above

14. Each edge of a cube is increased by 50%. What is the percent increase in the surface area of the cube?

(a) 50% **(b)** 125% **(c)** 150% **(d)** 300% **(e)** None of the above

15. If $f(x) = x^3 + 1$ and c is a positive number such that $f(c + 2) = f(c) + f(2)$ then c equals:

(a) $\sqrt{\frac{30}{3}} + 1$ **(b)** $\sqrt{\frac{7}{6}} + 1$ **(c)** $\sqrt{10} - 1$ **(d)** $\sqrt{\frac{7}{6}} - 1$ **(e)** None of the above

16. $\log_2(\log_3(\log_4 2^n)) = 2$. Find n.

(a) $\frac{27}{\log_4 2}$ **(b)** $\frac{4^{81}}{2}$ **(c)** 162 **(d)** 24 **(e)** None of the above

17. A square and an equilateral triangle both have perimeter of 7 cm. How much bigger is the area of the square than the area of the triangle?

(a) $\frac{49}{2}\left(\frac{1}{8} - \frac{\sqrt{3}}{9}\right)$ cm^2 **(b)** $\frac{49}{16} - \frac{49}{36}$ cm^2 **(c)** $\frac{49}{4}\left(\frac{1}{4} - \frac{\sqrt{3}}{9}\right)$ cm^2

(d) $\frac{49}{16} - \frac{49}{18}$ cm^2 **(e)** None of the above

18. The operations of addition and multiplications of sets and sets and sets and numbers are defined in the following way: if A and B are sets of numbers and c is a number,

$$\begin{aligned} A + B &= \{x + y | x \in A \text{ and } y \in B\} \\ A \cdot B &= \{x \cdot y | x \in A \text{ and } y \in B\} \\ A + c &= \{x + c | x \in A\} \\ c \cdot A &= \{c \cdot x | x \in A\} \end{aligned}$$

Let $E = \{0, \pm 2, \pm 4, \dots\}$ be the set of all even integers. Which of the following statements is false?

(a) $E + 1 = E + (-1)$ **(b)** $E + E = 2 \cdot E$

(c) $E = -1 \cdot E$ **(d)** More than one of them are false

(e) None of the statements is false

19. How many triangles with positive area are there whose vertices are points in the xy-plane with integer coordinates (x, y) satisfying $1 \le x \le 4$ and $1 \le y \le 4$?

(a) 496 **(b)** 500 **(c)** 512 **(d)** 516 **(e)** 560

20. Two copies of the parabola with equation $y = x^2$ are modified in the following ways:

    - The first copy is scaled in the vertical direction by a factor of 2.
    - The second copy is scaled in the horizontal direction by a positive factor k.

    The resulting graphs are exactly the same. Find k.

    (a) $k = 2$ **(b)** $k = \sqrt{2}$ **(c)** $k = \frac{1}{2}$ **(d)** $k = \frac{\sqrt{2}}{2}$ **(e)** None of the above

21. Suppose n and b are positive numbers. If $\log_b n = 2$ and $\log_n 2b = 2$, what is b?

    (a) 1 **(b)** $\sqrt{2}$ **(c)** $\sqrt[3]{4}$ **(d)** $\sqrt[3]{2}$ **(e)** None of the above

22. Which of the following is not equal to $\sec^2\theta \csc^2\theta$?

    (a) $\sec^2\theta + \csc^2\theta$

    (b) $\dfrac{\tan^2\theta + 1}{1 - \cos^2\theta}$

    (c) $(\tan\theta + \cot\theta)^2$

    (d) $\dfrac{1}{\sec^2\theta} + \dfrac{1}{\csc^2\theta}$

    (e) More than one of them are not equal to $\sec^2\theta \csc^2\theta$

23. What is $(1 + i)^{13}$?

    (a) $-64 - 64i$ **(b)** $-8192i$ **(c)** $64\sqrt{2} + 64\sqrt{2}i$ **(d)** $-12 - 12i$

    (e) None of the above

24. Juniors and Seniors from four schools are going to a State competition. There are 40 students from Atherton High School, 25 from Bayside High, 20 from Clairemont High School and 15 from Devonshire High School. 16 of the students from Atherton, 5 of the students from Bayside, 15 of the students from Clairemont, and 9 from Devonshire, are Juniors. If a Senior is selected at random, what is the probablilty that he or she is from Clairemont?

    (a) .05 **(b)** .25 **(c)** $\frac{1}{11}$ **(d)** $\frac{1}{3}$ **(e)** None of the above

25. Suppose $f(x) = f\left(\frac{1}{x}\right)$, for all $x \neq 0$, and that $f(a) + f(b) - f(a + b) = \frac{2(a^3 - b^3)}{ab(a^2 - b^2)}$ for all a and b such that $a^2 \neq b^2$. Suppose also that $f(2) = 5$. What is $f(2.5)$?

    (a) 4.2 **(b)** 5.8 **(c)** 7.9 **(d)** 1 **(e)** None of the above