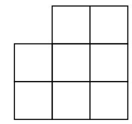
Saginaw Valley State University

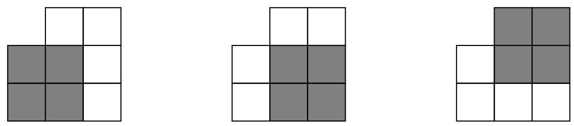
2008 Math Olympics – Level I Solutions

1. How many squares are there altogether in this diagram?



(a) 8 **(b)** 9 **(c)** 10 **(d)** 11 **(e)** 12

SOLUTION **(d)**: There are 8 small (1×1) squares. In addition to that there are 3 large (2×2) squares:



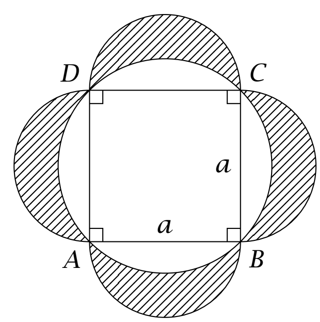
There are no larger squares than 2×2.

2. How many different ways are there to pay a \$9.75 bill if only dimes and quarters are available?

(a) 39 **(b)** 19 **(c)** 20 **(d)** 40 **(e)** None of the above

SOLUTION **(c)**: We need to find the number of non-negative integer solutions of the equation $10x+25y = 975$, or $2x + 5y = 195$, or $2x = 195 - 5y$. Because the right hand side is divisible by 5, x must also be divisible by 5, so $x = 5d$ for some non-negative integer d. Then the equation becomes $10d = 195 - 5y$ or $2d = 39 - y$. So the number of solutions will be the number of non-negative even integers less than or equal to 39. There are $\frac{39+1}{2} = 20$ such numbers.

3. The square $ABCD$, with a side a, is inscribed in a circle. Outside of each side of the square, there are semicircles as shown. What is the area of the shaded region?



(a) $\dfrac{\pi a^2}{2}$

(b) a^2 **(c)** $\pi a^2\left(1 - \dfrac{\sqrt{2}}{2}\right)$

(d) $a^2\left(1 - \dfrac{\sqrt{2}}{2}\right)$ **(e)** None of the above

SOLUTION **(b)**:

$$\text{Shaded area} = A_{✿} - A_{\bigcirc} = a^2 + 4\left(\frac{1}{2}\pi\left(\frac{a}{2}\right)^2\right) - \pi\left(\frac{\sqrt{2}a}{2}\right)^2 = a^2 + \frac{\pi a^2}{2} - \frac{\pi a^2}{2} = a^2$$

4. The owners of an apartment complex find that if they charge \$500 per month for rent they can rent out all 200 units, but for every \$10 increase in rent, they lose one renter. How much should they charge per month to maximize the monthly income?

   (a) \$125 **(b)** \$1250 **(c)** \$2500 **(d)** \$500 **(e)** None of the above

   SOLUTION **(b)**: Let r be the rent per unit, and let n be the number of units. The relation between r and n is given by $-10(n-200) = r - 500$, or $n = -\frac{1}{10}(r-500)+200 = -\frac{1}{10}r + 250$. The total monthly income is $n \cdot r = \left(-\frac{1}{10}r + 250\right) \cdot r$, which is a quadratic function of r whose graph is a downward facing parabola with r-intercepts at 0 and 2500. The highest point on the parabola is the vertex, whose r-coordinate is 1250.

5. In a small school 7 students are in the Math class, 7 are in the Science class, and 8 are in the Music class. Three students are in both Math and Science, five are in both Science and Music, and four are in both Math and Music. Two students are taking all three classes. How many students are there taking at least one of the three classes?

   (a) 10 **(b)** 12 **(c)** 22 **(d)** 36 **(e)** None of the above

   SOLUTION **(b)**: Adding the number of students in each class gives us $7 + 7 + 8 = 22$. However, the students that are taking two classes were counted twice, and the students that are taking all three classes were counted three times, so first we need to subtract the students that are taking at least two classes: $22 - 3 - 5 - 4 = 10$. Now the students who are taking all three classes, and who were originally counted three times, were subtracted three times, so now they are not counted at all. So we need to add their number to the sum again: $10 + 2 = 12$.

6. If the parabola $y = x^2 + bx + c$ has an x-intercept at 2 and a y-intercept at -8, what is its vertex?

   (a) $(-1,-9)$ **(b)** $(-1,-5)$ **(c)** $(1,-5)$ **(d)** $(1,9)$ **(e)** None of the above

   SOLUTION **(a)**: Since -8 is the y-intercept we have $0^2 + b \cdot 0 + c = -8$ so $c = -8$. Next since 2 is an x-intercept we have $2^2 + b \cdot 2 - 8 = 0$ so $b = 2$ and the parabola is $y = x^2 + 2x - 8$. By completing the square or using the vertex formula we find that the vertex is $(-1,-9)$.

7. The product of $\sqrt{5}$ and $\sqrt[3]{7}$ is

   (a) $\sqrt[5]{35}$ **(b)** $\sqrt[6]{35}$ **(c)** $\sqrt[5]{6125}$ **(d)** $\sqrt[6]{6125}$ **(e)** None of the above

   SOLUTION **(d)**: Let $x = \sqrt{5} \cdot \sqrt[3]{7}$. Then

   $$x^6 = \left(\sqrt{5} \cdot \sqrt[3]{7}\right)^6 = 5^3 \cdot 7^2 = 6125$$

   so $x = \sqrt[6]{6125}$.

8. A circular table has exactly 60 chairs around it. There are N people seated around the table. The next person coming to the table will have to be seated next to an occupied seat. Find the smallest possible value of N.

(a) 15 **(b)** 20 **(c)** 30 **(d)** 40 **(e)** 58

Solution **(b)**: For the next person to have to sit next to an occupied seat, there cannot be three consecutive chairs currently unoccupied (otherwise the next person would simply sit in the middle of the three empty chairs). Therefore for every three consecutive chairs at least one of them has to be occupied. Since we are looking for the smallest N, exactly one of the three will have to be occupied, and each two people will have to have two empty seats between them. Therefore the number of people sitting at the table is $1/3$ of the number of seats, or 20 people.

9. Simplify the following expression:

$$\frac{\frac{yz}{x} - \frac{xz}{y}}{\frac{x-2y}{yz} + \frac{y}{xz}}$$

(a) $\dfrac{(y-x)(y+x)}{y(x^2-2x+y)}$ **(b)** $-\dfrac{z^2}{2x}$ **(c)** 0 **(d)** $-\dfrac{z^2(x+y)}{(x-y)}$

(e) None of the above

Solution **(d)**:

$$\frac{\frac{yz}{z} - \frac{xz}{z}}{\frac{x-2y}{yz} + \frac{y}{xz}} = \frac{\frac{y^2z^2-x^2z^2}{xyz}}{\frac{x^2-2xy}{xyz} + \frac{y^2}{xyz}}$$
$$= \frac{y^2z^2 - x^2z^2}{x^2-2xy+y^2}$$
$$= \frac{-z^2(x-y)(x+y)}{(x-y)^2}$$
$$= -\frac{z^2(x+y)}{(x-y)}$$

10. For how many integers n between 1 and 100 can $x^2 + x - n$ be factored into the product of two linear factors with integer coefficients?

(a) 0 **(b)** 1 **(c)** 2 **(d)** 9 **(e)** 10

Solution **(d)**: Since the coefficient of x^2 is 1, and we require integer coefficients, it can only be factored as $(x-a)(x-b)$ where a and b are integers. Then $a+b=-1$ and $ab=-n$. Combining these two together, we get $a(a+1)=n$. Since $1 \le n \le 100$, a can be any integer starting with 1 and ending with 9, which gives us 9 different choices.

11. The strange operation $*$ is defined to be:

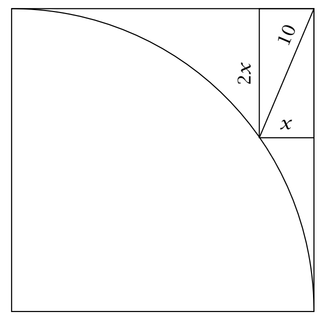
$$a * b := \frac{a}{a + \frac{1}{b}}$$

where a and b are real numbers. Find $5 * (1 * 3)$.

(a) $\frac{15}{19}$ **(b)** $\frac{5}{7}$ **(c)** $\frac{15}{16}$ **(d)** $\frac{20}{23}$ **(e)** None of the above

SOLUTION **(a)**: $5 * (1 * 3) = 5 * \left(\frac{1}{1+\frac{1}{3}}\right) = 5 * \frac{3}{4} = \frac{5}{5+\frac{4}{3}} = \frac{5}{\frac{19}{3}} = \frac{15}{19}$

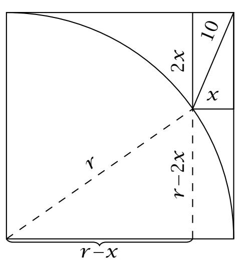
12. A quarter circle is inscribed in a square. A rectangle with dimensions x and $2x$ (as shown) is inscribed in the region of the square outside of the circle, touching the circle, as shown. What is the radius of the circle if the diagonal of the rectangle equals 10 units?



(a) $10\sqrt{5}$

(b) $2\sqrt{5}$ **(c)** 20 **(d)** $6\sqrt{5}$ **(e)** None of the above

SOLUTION **(a)**:



First we will find x using Pythagorean theorem: $x^2 + 4x^2 = 100$, or $x^2 = 20$, or $x = 2\sqrt{5}$. Then, using Pythagorean theorem again, $r^2 = (r-x)^2 + (r-2x)^2 = 2r^2 - 6xr + 5x^2$. Pluging in $x = 2\sqrt{5}$, we get $0 = r^2 - 12\sqrt{5}r + 100$. Using quadratic formula,

$$r = \frac{12\sqrt{5} \pm \sqrt{720 - 400}}{2} = 6\sqrt{5} \pm 4\sqrt{5}$$

Obviously the solution $r = 2\sqrt{5} = x$ does not fit the described situation (the quarter circle is inscribed inside of the rectangle), so the correct solution must be $r = 10\sqrt{5}$.

13. The circumference of Gus's pizza is 30% greater than the circumference of Max's pizza. The area of Gus's pizza is what % greater than the area of Max's pizza?

(a) 30% **(b)** 60% **(c)** 69% **(d)** 90% **(e)** None of the above

SOLUTION **(c)**: Let R = the radius of Gus's pizza and r = the radius of Max's pizza. Then $2\pi R = 1.3(1\pi r)$ so $R = 1.3r$. Therefore the area of Gus's pizza $= \pi R^2 = \pi(1.3r)^2 = 1.69\pi r^2 = 1.69 \cdot$ (area of Max's pizza).

So the area of Gus's pizza is 69% greater than the area of Max's pizza.

14. If $xy = 10$ and $x^2y + xy^2 + x + y = 99$, find $x^2 + y^2$.

(a) 61 **(b)** 71

(c) 81 **(d)** Not enough information given

(e) None of the above

Solution **(a)**:

$$\begin{aligned} x^2y + xy^2 + x + y &= 99 \\ 10x + 10y + x + y &= 99 \\ 11x + 11y &= 99 \\ x + y &= 9 \end{aligned}$$

Then

$$\begin{aligned} (x+y)^2 &= x^2 + 2xy + y^2 \\ 81 &= x^2 + y^2 + 20 \\ x^2 + y^2 &= 61 \end{aligned}$$

15. The digits of the whole numbers from 1 to 99 are concatenated in order to form thye number N:

$$N = 1234567891011121314\ldots979899$$

Which of the following is true?

(a) N is divisible by 3 but not by 6 and 9 **(b)** N is divisible by 3 and 6 but not by 9

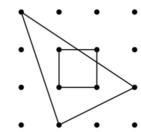
(c) N is divisible by 3 and 9 but not by 6 **(d)** N is not divisible by any of 3, 6 or 9

(e) None of the above

Solution **(c)**: The sum of the digits on N is $10(1+2+3+\cdots+9)+10(1+2+3+\cdots+9) = 20\cdot 45 = 900$ which is divisible ny 3 and 9, so N is divisible by both 3 and 9.

N is not divisible by 2 since it ends in 9, so N cannot be divisible by 6.

16. The dots are one unit apart. What is the area (in square units) of the region that is common to both the square and the triangle?



(a) $\frac{9}{10}$ **(b)** $\frac{15}{16}$ **(c)** $\frac{8}{9}$ **(d)** $\frac{11}{12}$ **(e)** $\frac{14}{15}$

SOLUTION **(d)**: The area of the common region is the area of the square minus the area of the little triangle that lies inside the square and outside of the large triangle. The line that forms the hypotenuse of the little triangle has slope $-\frac{2}{3}$. To find the intersection point of this line with the top side of the square, consider this:

From the upper left corner of the figure to the intersection point, the vertical coordinate decreases by one unit. With every two units the vertical coordinate decreases, the horizontal coordinate increases by three units. This means that the horizontal coordinate of the intersection point is $1,5$ units larger than the horizontal coordinate of the upper left corner of the figure. Therefore the intersection point is exactly in the middle of the upper side of the square.

When moving half unit to the right along the slanted line, we travel $1/3$ unit down. Therefore the legs of the little right triangle are $1/2$ unit and $1/3$ unit, which makes the area of the little triangle

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12}$$

The area of the common region is then

$$1 - \frac{1}{12} = \frac{11}{12}$$

17. A used car salesman sells 2 cars for \$6000 each. He makes a 25% profit on the first car and takes a 20% loss on the second car. Which of the following is true?

(a) The salesman has a net gain of 5% on the two transaction combined.

(b) The salesman has a net gain of \$300 on the two transaction combined.

(c) The salesman has a net loss of \$300 on the two transaction combined.

(d) There is not enough information given to determine his net gain or loss.

(e) None of the above

Solution **(c)**: Let the first car cost be x and the second car cost be y. Then

$$\begin{aligned} 1.25x &= 6000 \\ \frac{5}{4}x &= 6000 \\ x &= \frac{4}{5} \cdot 6000 \\ x &= 4800 \end{aligned}$$

and

$$\begin{aligned} .8y &= 6000 \\ \frac{4}{5}y &= 6000 \\ y &= \frac{5}{4} \cdot 6000 \\ y &= 7500 \end{aligned}$$

The car salesman's total cost is $4800 + 7500 = 12,300$, his total revenue is $2 \cdot 6000 = 12,000$, which makes his net loss \$300.

18. Joe wants to cook for his friends using a recipe from a cookbook. He needs to feed a lot of people, so he triples the recipe. At the dinner only two thirds of his dish is eaten. Then as he is cleaning up he drops one fourth of the left overs on the floor, but he keeps the rest. If the original cookbook recipe makes 8 servings, how many servings will the left overs make?

(a) 2 **(b)** 4 **(c)** 6 **(d)** 8 **(e)** None of the above

Solution **(c)**: Joe makes $3 \cdot 8 = 24$ servings. After dinner there are $\frac{1}{3} \cdot 24 = 8$ servings remaining. After he drops some, he has $\frac{3}{4} \cdot 8 = 6$ servings left.

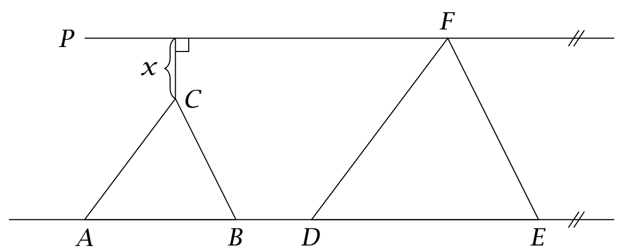
19. Max and Gus start their new jobs on the same day. In Max's schedule, one rest-day follows every three work-days. In Gus's schedule, three rest-days follow every seven work-days. How many times during their first 1000 days do they both have a rest-day on the same day?

(a) 48 **(b)** 50 **(c)** 72 **(d)** 75 **(e)** 100

SOLUTION **(e)**: Max follows a 4-day cycle, while Gus follows a 10-day cycle. Since the least common multiple of 4 and 10 is 20, together they follow a 20-day cycle. There are 50 such cycles in the 1000 days.

During each 20-day cycle, Max rests on every fourth day, while Gus rests on the days 8, 9 10, 18, 19 and 20. They have two common rest days in each 20-day cycle. The number of common rest days in the first 1000 days is $50 \cdot 2 = 100$.

20. Two similar triangles, $\triangle ABC$ and $\triangle DEF$ have bases on line AE. The area of $\triangle ABC$ is 100 square units, while the area of $\triangle DEF$ is 144 square units. The side DE is 6 units long. The lines AE and PF are parallel. Find x.



(a) $14.\overline{6}$ **(b)** 28

(c) 19.2 **(d)** 8 **(e)** None of the above

SOLUTION **(d)**: The ratio of similarity of the two triangles is $\sqrt{\frac{144}{100}} = 1.2$. The height of $\triangle DEF$ perpendicular to DE is $144/(\frac{1}{2} \cdot 6) = 48$. The height of $\triangle ABC$ perpendicular to AB is $\frac{1}{1.2} \cdot 48 = 40$, so $x = 48 - 40 = 8$.

21. How many ordered pairs (m, n) of positive integers are solutions to

$$\frac{4}{m} + \frac{2}{n} = 1?$$

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** None of the above

SOLUTION **(d)**: Since m and n are supposed to be positive, both $\frac{4}{m}$ and $\frac{2}{n}$ must be less than 1, and so $m > 4$ and $n > 2$.

$$\begin{aligned} \frac{4}{m} + \frac{2}{n} &= 1 \\ 4n + 2m &= mn \\ 0 &= mn - 4n - 2m \\ 8 &= mn - 4n - 2m + 8 \\ 8 &= (m-4)(n-2) \end{aligned}$$

We are looking for the number of ways 8 can factored into a product of two positive integers: $8 = 8 \cdot 1 = 4 \cdot 2 = 2 \cdot 4 = 1 \cdot 8$. These will correspond to $(m, n) = (12, 3)$, $(m, n) = (8, 6)$, $(m, n) = (6, 6)$ and $(m, n) = (5, 10)$.

22. A bag contains a mix of red and blue marbles. If one red marble is removed, then one-seventh of the remaining marbles are red. If two blue marbles are removed instead of one red, then one-fifth of the remaining marbles are red. How many marbles were in the bag originally?

(a) 8 **(b)** 22

(c) 36 **(d)** 57

(e) Not enough information given

SOLUTION **(b)**: Let r be the initial number of red marbles in the bag, and let n be the total number of marbles the bag contained initially. Then

$$\frac{r-1}{n-1} = \frac{1}{7} \quad \text{and} \quad \frac{r}{n-2} = \frac{1}{5}$$

This will give us a system of equations:

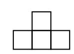
$$\begin{cases} 7r - 7 = n - 1 \\ 5r = n - 2 \end{cases}$$

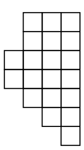
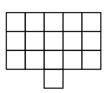
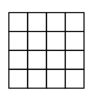
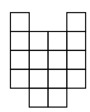
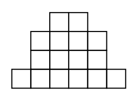
or

$$\begin{cases} 7r - n = 6 \\ 5r - n = -2 \end{cases}$$

Solving this system will give us $r = 4$ and $n = 22$.

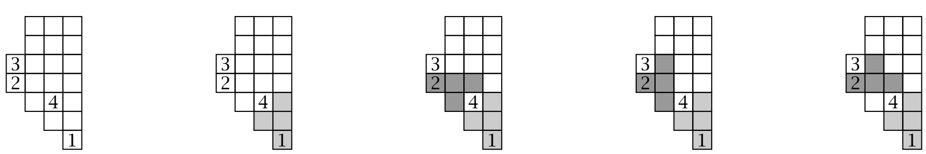
23. Which shape *cannot* be filled, without any overlapping, using copies of the tile showed on the right?



(a)  **(b)**  **(c)**  **(d)**  **(e)** 

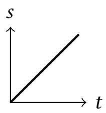
SOLUTION **(a)**: The following shapes can be filled as required:

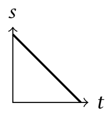
To show that the remaining shape cannot be filled, we will number some of the squares:

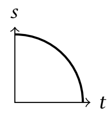


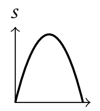
The only way to cover the square 1 is by placing the tile as shown on the second picture above. To cover the square 2, we can place the tile in one of the three positions shown above. In each of these positions, however, at least one of the squares 3 and 4 cannot be covered. Therefore this shape cannot be filled by copies of the given tile without overlapping.

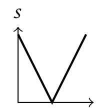
24. I throw a ball vertically up into the air. Which of these graphs might reasonably show the speed s of the ball as a function of time t since leaving my hand?

(a) 

(b) 

(c) 

(d) 

(e) 

Solution **(e)**: After the ball is thrown into the air, it will first travel up, while slowing down the whole time. Therefore the speed will initially be decreasing. When the speed reaches 0, the ball will stop momentarily, and then it will start falling down again, gaining speed as it falls. Which means that after reaching 0, the speed will start increasing again. The only graph that reasonably represents this behavior is the last one.

25. If you travel 60 miles at 20 miles per hour and then travel 60 miles at 30 miles per hour, what is your average speed for the entire trip?

(a) 60 mph **(b)** 25 mph **(c)** 23 mph **(d)** 50 mph **(e)** None of the above

Solution **(e)**: The total distance is 120 miles. The time required for the first 60 miles is $60 \div 20 = 3$ hours, and the time required for the next 60 miles is $60 \div 30 = 2$ hours. So the total time is 5 hours. The average speed is $120 \div 5 = 24$ miles per hour. None of the provided answers is correct.