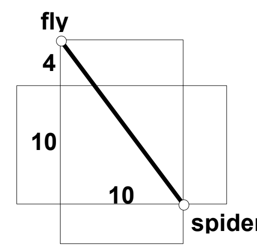
4/12/02 **2002 Math Olympics – Level II Solutions**

1) **(D)** The volume of a cylinder is $\pi r^2 h$, so if the radius is increased by 50% to $r + .5r = 1.5r$, then the volume is $\pi(1.5r)^2 h = 2.25\pi r^2 h$. Therefore, the percent increase is $\dfrac{2.25\pi r^2 h - \pi r^2 h}{\pi r^2 h} = 1.25$ or 125%.

2) **(B)** The log function has domain $(0,\infty)$. The domain of $y = \log x^2$ is $(-\infty,0)\cup(0,\infty)$ since x^2 is positive on this interval, but the domain of $y = 2\log x$ is $(0,\infty)$ since this is the interval on which x is positive. Therefore, B is true and A, C, and D must be false – they cannot be the same function, or multiples of each other, or have the same domain if their domains are different.

3) **(D)** Tracy calculated $T/6$ and the correct average is $T/5$. Since the correct average is 14 more than what she calculated, we have $T/6 + 14 = T/5$.

4) **(D)** Looking at the net for the room, the shortest distance (d) is along the straight line. By the Pythagorean Theorem, $d = \sqrt{14^2 + 10^2} = \sqrt{296}$. Spiders can't fly.



5) **(A)** Let d be the diameter of the balls. The triangle shown is a 30-60-90 triangle since one leg is half the length of the hypotenuse. So, the side opposite the 60 degree angle is $\sqrt{3}$ times the short leg. This gives $33 - d = 2d\sqrt{3}$ or $d = \dfrac{33}{1+2\sqrt{3}}$. Rationalizing, we get $d = \dfrac{33(1-2\sqrt{3})}{-11} = 6\sqrt{3} - 3$.

6) **(C)** Let a_n represent the nth term of the sequence. We have $a_1 = a_7 = 8$ and, being the sum of the previous two terms, $a_3 = a_1 + a_2 = 8 + a_2$. Continuing, $a_4 = a_2 + a_3 = 8 + 2a_2$, $a_5 = a_3 + a_4 = 16 + 3a_2$, $a_6 = a_4 + a_5 = 24 + 5a_2$, $a_7 = a_5 + a_6 = 40 + 8a_2$. Now, substituting 8 for a_7 in the last equation we have $8 = 40 + 8a_2$, or $a_2 = -4$. Substituting -4 in the expression for a_5, $a_5 = 16 + 3(-4) = 4$.

7) **(D)** If $b^a = 1$ then one of the following three cases must be true.

I. $a = 0 \quad and \quad b \neq 0$: Solving $x^2 - 9x + 20 = 0$, $(x-4)(x-1) = 0$ or $x = 4, 5$. Checking, $x^2 - 5x + 5 \neq 0$ at 4 or 5.

II. $b = 1$: Solving $x^2 - 5x + 5 = 1 \Leftrightarrow x^2 - 5x + 4 = 0$, so $(x-4)(x-1) = 0$ or $x = 1, 4$.

III. $b = -1 \quad and \quad a \quad is\ even$: Solving, $x^2 - 5x + 5 = -1 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow (x-6)(x+1) = 0$, so $x = -1, 6$. Check that $(-1)^2 - 9(-1) + 20 = 30, 6^2 - 9(6) + 20 = 2$ are both even.

Summing all the different values, $4 + 5 + 1 + (-1) + 6 = 15$.

8) **(C)** Since 202 base 5 is $2(5^2) + 0(5^1) + 2(5^0) = 52$ in base 10, the given base 3 numeral must be equal to 52 base 10. So, $52 = 1(3^3) + x(3^2) + 2(3^1) + 1(3^0)$, or, equivalently, $52 = 9x + 34$, Solving, $x = 2$.

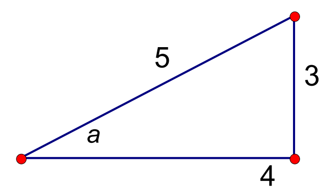
9) **(D)** Writing each with the common base 2, the given expression is equivalent to

$$(\log_2 3)\left(\frac{\log_2 4}{\log_2 3}\right)\left(\frac{\log_2 5}{\log_2 4}\right)\left(\frac{\log_2 6}{\log_2 5}\right)\left(\frac{\log_2 7}{\log_2 6}\right)\left(\frac{\log_2 8}{\log_2 7}\right) = \log_2 8 = 3$$

10) **(C)** Each being a factor, and the leading coefficient of f being a, $ax^3 + bx^2 + cx + 3 = a(x-2)(x-3)(x-4)$. Equating the constant term on each side, $3 = a(-24)$, so $a = {-1}/{8}$.

11) **(C)** Using the right triangle ratios and the Pythagorean theorem to find the third side of the right triangle, $\sin a = 3/5$ implies $\cos a = \pm 4/5$. But, $\cos a < 0$, so $\cos a = -4/5$ and

$\sin(2a) = 2 \sin a \cos a = 2\left(3/5\right)\left(-4/5\right) = -24/25$



12) **(B)** Average speed is total distance divided by the total time. If we let d be the distance for one lap, then the total distance is $2d$. The total time is the sum of the times for each lap, which is $d/180 + d/120 = 5d/360$. Therefore, the average speed is $\dfrac{2d}{\frac{5d}{360}} = \frac{2}{5}360 = 144$ mph.

13) **(A)**Since $\tan\theta = {}^{-3}\!/_4 < 0$ and $\sec\theta > 0$, we know $\cos\theta > 0$ and $\sin\theta < 0$, so $\theta = \arctan(-{}^{3}\!/_4)$ is a fourth quadrant angle. Then, $\arctan({}^{3}\!/_4)$ is the first quadrant angle $-\theta$. We have $\sin(\arctan({}^{3}\!/_4)) = \sin(-\theta) = -\sin\theta$, and from the same triangle as in (11) above, $\sin\theta = -{}^{3}\!/_5$ since in the fourth quadrant. Its negative is ${}^{3}\!/_5$.

14) **(C)** Using N for nickel, D for dime, and Q for quarter, the possible change combinations are 10N, 5D, 2Q, 8N&1D, 6N&2D, 4N &3D, 2N&4D, 1Q5N, 1Q1D3N, 1Q2D1N. Of these ten possible outcomes, three have 1 quarter.

15) **(B)**There are 3 possible positions that John and Julie can occupy so that they are not next to each other: X _ _ X, X _ X _, or _ X _ X. For each of these three positions, John and Julie can be in either order, and the other two can be in either order in the other slots. So, there are 3(2)(2)=12 possible ways.

16) **(C)** At 4:00 p.m. the hour hand is 20 minutes ahead of the minute hand. If we let x be the number of minutes it will take the minute hand to catch the hour hand, then the hour hand will travel $x-20$ minutes. If r is the speed of the hour hand, then $12r$ is the speed of the minute hand. The time it will take for them to coincide is then $\frac{x}{12r} = \frac{x-20}{r}$, which is equivalent to $11xr = 240r$, or $x = \frac{240}{11} = 21\frac{9}{11}$ minutes, which is closest to 4:22 p.m.

17) **(C)**Avoiding 0 and repeats, there are $9\cdot 8\cdot 7\cdot 6 = 3024$ weird four-digit numbers. A 4-digit goofy number could have 2, 4, 6, or 8 as the first digit, and any of 0, 2, 4, 6, 8 for the remaining three digits, so there are $4\cdot 5\cdot 5\cdot 5 = 500$ of these. We need to throw out the numbers that are both weird and goofy, as these were counted twice. If weird and goofy, each digit must be 2, 4, 6, or 8 and there can be no repeating digits, so there are $4\cdot 3\cdot 2\cdot 1 = 24$ of these. There are, then, 3024+500-24=3500 that are weird or goofy.

18) **(D)**Writing down the first few terms and expressing them in terms of a_1, a_2 and a_3:

$a_4 = a_3(a_2 + a_1),\quad a_5 = a_4(a_3 + a_2) = a_3(a_2 + a_1)(a_3 + a_2)$ and

$a_6 = a_5(a_4 + a_3) = a_3(a_2 + a_1)(a_3 + a_2)[a_3(a_2 + a_1) + a_3] = a_3(a_2 + a_1)(a_3 + a_2)[a_3(a_2 + a_1 + 1)]$. Therefore, $144 = a_3^2(a_3 + a_2)(a_2 + a_1)(a_2 + a_1 + 1)$. This shows that a_3^2 is a factor and so are the consecutive integers $(a_2 + a_1)$ and $(a_2 + a_1 + 1)$. Now $144 = 12^2 = 2^4\cdot 3^2$, so it has factor 1,2,3,4,6,8,9,12,16,18,24,36,48,72,144. The only consecutive pairs are 1,2 or 2,3 or 3,4 or 8,9.

I. (1,2): Can't occur since $a_1 + a_2 \geq 2$.

II. (2,3): If $a_1 + a_2 = 2$ then they are both 1, so $144 = 6a_3^2(a_3 + 1)$ or $24 = a_3^2(a_3 + 1)$. No factor a_3 of 24 satisfies this criteria.

III. (3,4): If $a_1 + a_2 = 3$ then $144 = 12a_3^2(a_3 + a_2)$ or $12 = a_3^2(a_3 + a_2)$. Therefore, $a_3 = 2$ and $a_2 = 1$, which implies $a_1 = 2$. Substituting, $a_4 = 6, a_5 = 18,$ and $a_7 = 144(18 + 6) = 3456$.

IV. (8,9): If $a_1 + a_2 = 8$ then $144 = 72a_3^2(a_3 + a_2)$, so $a_3 = a_2 = 1$ leaving $a_1 = 7$. Substituting, $a_4 = 8, a_5 = 16,$ and $a_7 = 144(16 + 8) = 3456$.

19) **(D)** Following the steps: (1) Let a = age (2) $2a$ (3) $2a+5$ (4) $50(2a+5)$ (5) $50(2a+5)-365$ (6) $50(2a+5)-365+h$, where h = height in inches (7) $50(2a+5)-365+h+115$. Solving $50(2a+5)-365+h+115$ =6364, we have $100a+h=6364$. Safely assuming that Jim's grandfather is not taller than 100 inches(8ft 4in), we have $a=63$ years and $h=64$ inches.

20) **(B)** Let $y=x^2-1$, switch x and y to get $x=y^2-1$. Solving for y we get $y=\pm\sqrt{x+1}$. The domain of $f(x)$ is $(-\infty,0]$, so the range of $f^{-1}(x)$ is the same. So, $f^{-1}(x)=-\sqrt{x+1}$.

21) **(B)** The leading coefficient in the numerator and denominator is 1, so the horizontal asymptote is the line $y=1/1=1$. To see if the graph intersects it, then, we solve $1=\dfrac{x^3+2x+2}{x^3+x^2+x}$, or $x^3+x^2+x=x^3+2x+2$. This is equivalent to $x^2-x-2=(x-2)(x+1)=0$. So, $x=2,-1$.

22) **(B)** Equating the amount of acid, $10(.20)+40(.15)=50(x)$ or $x=8/50=0.16$.

23) **(B)** We need $4+3x-x^2>0$, or $x^2-3x-4=(x-4)(x+1)<0$. This parabola, $y=x^2-3x-4$ opens up, so its graph is negative (below the x-axis) between its zeros, which are -1 and 4.

24) **(A)** Let $f(x)=Ax^2+Bx+C$, then $f(1)=A+B+C<0$, so the point on the parabola with x coordinate 1 is below the x - axis. Since there are no real zeros this means the entire parabola must lie below the x axis. Therefore, it must open down, $A<0$, and the discriminant, B^2-4AC, must be negative. But $B^2-4AC<0$ and $A<0$ implies $C<0$.

25) **(B)** Here $\dfrac{1}{1-\cos t}-\dfrac{1}{1+\cos t}=\dfrac{1+\cos t-(1-\cos t)}{1-\cos^2 t}=\dfrac{2\cos t}{\sin^2 t}$. Solving $\sin^2 t+\cos^2 t=1$ for $\cos t$, we get $\cos t=\pm\sqrt{1-\sin^2 t}$. Since t is a third quadrant angle, the cosine function is negative, so $\cos t=-\sqrt{1-\sin^2 t}$.