2003 MATH OLYMPICS

LEVEL II

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1. $\left(\frac{1}{4}\right)^{-\frac{1}{4}} =$

   a. $\sqrt{2}$ b. $-\sqrt{2}$ c. 16 d. −16. e. None of the above

2. Let f be the function defined by $f(x) = ax^2 - \sqrt{2}$ for some positive a. If $f(f(\sqrt{2})) = -\sqrt{2}$, then a is

   a. $2-\sqrt{2}$ b. $\frac{\sqrt{2}}{2}$ c. $\frac{2+\sqrt{2}}{2}$ d. $\frac{2-\sqrt{2}}{2}$ e. None of the above

3. As a promotional campaign a music store gave away CDs. Four people received CDs. The first person got 1/3 of the CDs, the second person got 1/4 of the CDs, the third person got 1/6 of the CDs, and the fourth person got 12 CDs. How many CDs did the first person get?

   a. 8 b. 10 c. 14 d. 16 e. None of the above

4. Let f be a degree 3 polynomial with leading coefficient 1. Suppose that x=3 is a double zero of f and that $f(0)$=3. The third zero of f is

   a. −2 b. −1 c. −1/2 d. −1/3 e. None of the above

5. Four whole numbers, when added three at a time, give the sums 180, 197, 208, and 222. What is the largest of the four numbers?

   a. 77 b. 83 c. 89 d. 95 e. None of the above

6. If $f(x) = 1 - x^2$, find a constant c so that $\dfrac{f(a+h) - f(a)}{h} = c(2a+h)$

a. c=1 b. c=−1 c. c=2 d. c=−2 e. None of the above

7. Which of the following equations have the same graph?

I. $y = x - 2$ II. $y = \dfrac{x^2 - 4}{x + 2}$ III. $(x+2)y = x^2 - 4$

a. I and II only b. I and III only c. II and III only

d. I, II, and III e. None. All the equations have different graphs

8. A tank has two identical drainage valves. With only one valve open, the full tank can be drained in 4 hours. If we start draining the full tank using both valves until the tank is 1/4 full, then let the rest drain through one valve only, the time it will take to drain the tank is

a. 2.25 hours b. 2.5 hours c. 2.75 hours d. 3 hours e. None of the above

9. Find the integer a so that $\tan\left[\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)\right] + \tan\left[\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)\right] = \dfrac{a}{\sqrt{3}}$.

a. 1 b. 2 c. 3 d. 4 e. None of the above

10. If $0 \le x \le \dfrac{\pi}{2}$ and $\sin x = 0.1$, then $\left(\sin\dfrac{x}{2} - \cos\dfrac{x}{2}\right)^2 =$

a. 0.1 b. 0.4 c. 0.7 d. 0.9 e. None of the above

11. Which of the following trigonometric equations is false for all x?

a. $\sin x = \frac{2}{\sqrt{5}}$ b. $\tan x = -100$ c. $\sec x = \frac{\sqrt{3}}{4}$

d. $\tan^2 x + 1 = \sec^2 x$ e. None of the above

12. When $\pi < t < \frac{3\pi}{2}$, the value of the expression $\frac{1}{1-\cos t} - \frac{1}{1+\cos t}$ in terms of $\sin t$ equals

a. $\frac{-2(1-\sin^2 t)}{\sin^2 t}$ b. $\frac{-2\sqrt{1-\sin^2 t}}{\sin^2 t}$ c. $\frac{2\sqrt{1-\sin^2 t}}{\sin^2 t}$

d. $\frac{2(1-\sin^2 t)}{\sin^2 t}$ e. None of the above

13. The equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point (4,3) is

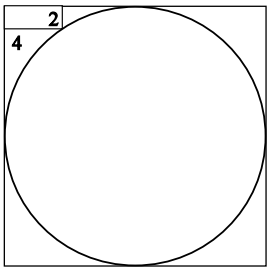
a. $3x + 4y = 24$ b. $3x - 4y = 0$ c. $4x - 3y = 7$

d. $4x + 3y = 25$ e. None of the above

14. How many positive integers less than 50 have an odd number of positive integer divisors?

a. 3 b. 5 c. 7 d. 9 e. None of the above

15. A circle is inscribed in a square. A 2×4 rectangle shares an upper left vertex with the square and its lower right vertex is on the circle as in the figure. What is the radius of the circle?



a. 4 b. 8 c. 10

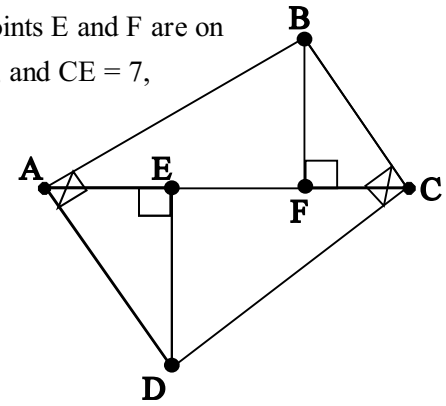
d. 12 e. None of the above

16. Two circles have the same radius *r*. Their centers are a distance of $r\sqrt{2}$ apart. The area of the intersection of the circles is

a. $\frac{\pi r^2}{4}$ b. $\left(2-\frac{\pi}{2}\right)r^2$ c. $\left(\frac{\pi}{2}-1\right)r^2$

d. $\frac{2\pi r^2}{3}$ e. None of the above

17. In the figure, ABCD is a quadrilateral with right angles at A and C. Points E and F are on $\overline{AC}$, and $\overline{DE}$ and $\overline{BF}$ are perpendicular to $\overline{AC}$. If AE = 3, DE = 5, and CE = 7, then BF =



a. 3.6 b. 4 c. 4.2

d. 4.5 e. None of the above

18. If $\log_a 2 = 3$ and $\log_2 x = -\frac{3}{2}$, then $\log_a\left(8\cdot\sqrt{\frac{x}{a}}\right)$ is

a. 6.25 b. 8 c. 8.25 d. 9.25 e. None of the above

19. If $\log_4 2a^{11} = 6$, then a equals

a. 2 b. 3 c. $\sqrt{2}$ d. -2 e. None of the above

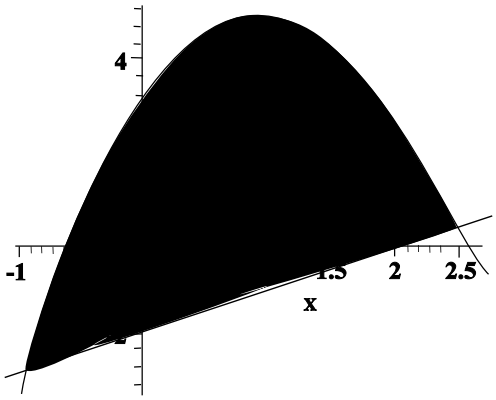
20. In how many ways can 6 people be lined up to get on a bus if 2 specific persons refuse to follow each other?

a. 240 b. 480 c. 600 d. 720 e. None of the above

21. Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta, and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can Professor Alpha, Beta, and Gamma choose their chairs?

a. 12 b. 36 c. 60 d. 84 e. None of the above

22. Find the maximum vertical distance d between the parabola $y = -2x^2 + 4x + 3$ and the line $y = x - 2$ throughout the shaded region in the figure.



a. $\frac{47}{8}$ b. $\frac{49}{8}$

c. $\frac{50}{8}$ d. $\frac{48}{8}$

e. None of the above

23. Two men stand 200 feet apart with a vertical flagpole in between them. If the angles of elevation of the top of the flagpole are 30° and 45°, how tall is the pole?

a. $\frac{200}{1+\sqrt{3}}$ b. $\frac{200\sqrt{3}}{1+\sqrt{3}}$ c. $\frac{200\sqrt{3}}{\sqrt{3}-1}$ d. $\frac{200}{\sqrt{3}-1}$ e. None of the above

24. At noon, a car and a van are 120 miles apart on a straight road and driving toward each other at a speed of 40 mi/hr. A fly starts from the front bumper of the van at noon and flies to the bumper of the car, then immediately back to the bumper of the van, back to the car, and so on, until the car and the van meet. If the fly flies at a speed of 100 mi/hr, what is the total distance it travels?

a. 150 miles b. 120 miles c. 100 miles d. 200 miles e. None of the above

25. An amoeba propagates by simple division; each split into two takes 3 minutes to complete. When such an amoeba is put into a glass container with a nutrient fluid, the container will be full of amoebas in one hour. How long would it take for the container to be filled if we start with not one amoeba, but two?

a. 30 min. b. 45 min. c. 51 min. d. 57 min. e. None of the above