Saginaw Valley State University

2022 Math Olympics — Level II Solutions

1. For $n \geq 2$, let k_n be the product $5 \cdot 10 \cdot 17 \cdot 26 \cdot \cdots \cdot (n^2+1)$, and let

$$S_n = \left(1 - \frac{1}{2^4}\right)\left(1 - \frac{1}{3^4}\right)\cdots\left(1 - \frac{1}{n^4}\right).$$

What is S_n/k_n?

(a) $\frac{1}{2}\frac{1}{(n!)^2}\left(1+\frac{1}{n}\right)$ **(b)** $\frac{1}{k_n}\frac{n^2}{n^4+1}$ **(c)** $\frac{n^4+1}{2n+2}\cdot\frac{1}{n!}$

(d) $\frac{1}{k_n}\frac{1}{n^4}\frac{1}{((n-1)!)^4}$ **(e)** None of the above

Solution **(a)**: Rewrite

$$\begin{aligned} S_n &= \left(1-\frac{1}{2^2}\right)\left(1+\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1+\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\left(1+\frac{1}{n^2}\right) \\ &= \left(1-\frac{1}{2^2}\right)\left(\frac{5}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(\frac{10}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\left(\frac{n^2+1}{n^2}\right) \end{aligned}$$

Dividing by k_n and canceling gives us

$$\begin{aligned} \frac{S_n}{k_n} &= \left(1-\frac{1}{2^2}\right)\left(\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\left(\frac{1}{n^2}\right) \\ &= \frac{1}{(n!)^2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right) \end{aligned}$$

because $2^2 \cdot 3^2 \cdots n^2 = (2 \cdot 3 \cdots n)^2 = (n!)^2$.
Using the difference of squares again,

$$\begin{aligned} \frac{S_n}{k_n} &= \frac{1}{(n!)^2}\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right) \\ &= \frac{1}{(n!)^2}\left(\frac{2-1}{2}\right)\left(\frac{2+1}{2}\right)\left(\frac{3-1}{3}\right)\left(\frac{3+1}{3}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{n+1}{n}\right) \\ &= \frac{1}{(n!)^2}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{n-1+1}{n-1}\right)\left(\frac{n-1}{n}\right)\left(\frac{n+1}{n}\right) \\ &= \frac{1}{(n!)^2}\left(\frac{1}{2}\right)\left(\frac{\cancel{3}}{\cancel{2}}\right)\left(\frac{\cancel{2}}{\cancel{3}}\right)\left(\frac{\cancel{4}}{\cancel{3}}\right)\left(\frac{\cancel{3}}{\cancel{4}}\right)\left(\frac{\cancel{5}}{\cancel{4}}\right)\left(\frac{\cancel{4}}{\cancel{5}}\right)\cdots\left(\frac{\cancel{n}}{\cancel{n-1}}\right)\left(\frac{\cancel{n-1}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right) \\ &= \frac{1}{(n!)^2}\left(\frac{1}{2}\right)\left(\frac{n+1}{n}\right) \end{aligned}$$

2. Consider the following equation:

$$3^x - 6^x = 12^x - \left(\frac{3}{2}\right)^x.$$

Which of the following is NOT a solution?

(a) $\frac{3\pi i}{\ln 2}$ **(b)** 0 **(c)** $\frac{6\pi i}{\ln 2}$ **(d)** $\frac{27\pi i}{\ln 2}$

(e) All are solutions

SOLUTION **(e)**: Rewrite

$$3^x - 6^x = 12^x - \left(\frac{3}{2}\right)^x$$

as

$$3^x - 6^x - 12^x + \left(\frac{3}{2}\right)^x = 0.$$

Dividing both sides by 3^x gives us

$$1 - 2^x - 4^x + \left(\frac{1}{2}\right)^x = 0.$$

We can then multiply by 2^x to get

$$2^x - 4^x - 8^x + 1 = 0$$

which can be written as

$$2^x - (2^x)^2 - (2^x)^3 + 1 = 0$$

The equation

$$u - u^2 - u^3 + 1 = 0$$

has two easy to guess solutions, $u = 1$ and $u = -1$. We can substitute 2^x for u:

– If $u = 1$, then $2^x = 1$. This can be written as

$$e^{x \ln 2} = 1$$

which is equaivalent to

$$x \ln 2 = 2k\pi i \text{ for any integer } k$$

Therefore any number x in the form $\frac{2k\pi i}{\ln 2}$ is a solution of the original equation.

– If $u = -1$, then $2^x = -1$. This can be written as

$$e^{x \ln 2} = -1$$

which is equaivalent to

$$x \ln 2 = (2k + 1)\pi i \text{ for any integer } k$$

Therefore any number x in the form $\frac{(2k+1)\pi i}{\ln 2}$ is a solution of the original equation.
Since any even integer can be written as $2k$ for some integer k, and any odd integer can be written as $2k+1$ for some integer k, all numbers in the form

$$\frac{n\pi i}{\ln 2}$$

are solutions of the original equation.

3. Consider the equation $p(x) : ax^2 + bx + c = 0$ whose coefficients a, b and c are all non-zero, and each of them satisfies an equation that results from removing the term containing that coefficent from the equation $p(x)$; for example, the coefficient b is a solution of the equation $ax^2 + c = 0$. What is the sum of all solutions of $p(x)$?

(a) Always 1 **(b)** Always -1 **(c)** Always 2 **(d)** 1 or -1 **(e)** 1 or 2

SOLUTION **(d)**: This is what we know about the coefficients a, b and c:

- a is a solution of $bx + c = 0$, so $ab + c = 0$.
- b is a solution of $ax^2 + c = 0$, so $ab^2 + c = 0$.
- c is a solution of $ax^2 + bx = 0$, so $ac^2 + bc = 0$.
- a, b and c are all non-zero.

From the first two equalities we get $ab = ab^2$, and since a and b are non-zero, we get $b = 1$.

- With $b = 1$, the first two equations both become $a + c = 0$
- The third equation becomes $ac^2 + c = 0$.

From these two equalities we get $ac^2 = a$ and since $a \neq 0$, we get $c^2 = 1$ or $c = \pm 1$. Since $a + c = 0$, $a = -c = \mp 1$.
That gives us only two possibilities for $p(x)$:

- One is $p(x) : x^2 + x - 1 = 0$. The solutions are

$$\frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

  The sum of these two solutions is -1.

- The other is $p(x) : -x^2 + x + 1 = 0$. Then the solutions are

$$\frac{-1 \pm \sqrt{1+4}}{-2} = \frac{-1 \pm \sqrt{5}}{-2}$$

  The sum of these two solutions is 1.

Alternatively, instead of discussing the two possibilities for $p(x)$, we can rewrite the equation as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. Since $\frac{c}{a} = -1 < 0$, the equation must have two real solutions. The sum of the two solutions is $-\frac{b}{a} = \pm 1$.

4. Find the sum of the first five terms of the series

$$\left\{\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots\right\}.$$

(a) $\dfrac{64}{81}$ **(b)** $\dfrac{550}{243}$ **(c)** $\dfrac{32}{81}$ **(d)** $\dfrac{422}{243}$ **(e)** None of the above

SOLUTION **(d)**: The terms of this series form a geometric sequence with first term $a_0 = \frac{2}{3}$ and common ratio $r = \frac{2}{2}$. Then the sum of the first five terms of the series is

$$s_5 = \frac{a_0(1 - r^5)}{1 - r} = \frac{\frac{2}{3}\left(1 - \left(\frac{2}{3}\right)^5\right)}{1 - \frac{2}{3}} = \frac{\frac{2}{3}\left(\frac{3^5 - 2^5}{3^5}\right)}{\frac{1}{3}} = 2\left(\frac{3^5 - 2^5}{3^5}\right).$$

Now $3^5 = 3^4 \cdot 3 = (3^2)^2 \cdot 3 = 9^2 \cdot 3 = 81 \cdot 3 = 243$. Similarly, $2^5 = 2^4 \cdot 2 = 16 \cdot 2 = 32$. We have

$$2\left(\frac{243 - 32}{243}\right) = 2\left(\frac{211}{243}\right) = \frac{422}{243}$$

5. Suppose f is a function such that $f(n+1) = f(n) + 3f(n-1)$, for all integers n. If $f(4) = 151$ and $f(1) = 1$, what is $f(-1)$?

(a) 12 **(b)** 4 **(c)** $-\frac{11}{3}$ **(d)** $\frac{13}{3}$ **(e)** None of the above

SOLUTION **(c)**: $f(4) = f(3) + 3f(2)$.
$f(2) = f(1) + 3f(0) = 1 + 3f(0)$ and $f(3) = f(2) + 3f(1) = 1 + 3f(0) + 3 \cdot 1 = 4 + 3f(0)$.
Putting these into $f(4)$ and simplifying we get
$151 = 4 + 3f(0) + 3(1 + 3f(0)) = 7 + 12f(0)$.
So $144 = 12f(0)$ and $f(0) = 12$. Then since $f(1) = f(0) + 3f(-1)$ we get $f(-1) = (f(1) - f(0))/3 = (1 - 12)/3 = -11/3$.

6. Rationalize the denominator:

$$\frac{1}{2^{5/6}+2^{1/2}+2^{1/3}+1}$$

(a) $\frac{1}{6}\left(2^{1/2}+2^{1/3}-2^{1/6}\right)$ **(b)** $\frac{1}{3}\left(2^{1/2}-1\right)\left(2^{2/3}-2^{1/3}+1\right)$

(c) $\frac{1}{2}\left(2^{7/6}-2^{5/6}+1\right)$ **(d)** $\left(2^{1/2}-1\right)\left(2^{2/3}-2^{1/3}+1\right)$

(e) None of the above

SOLUTION **(b)**: We can factor the denominator by grouping:

$$\begin{aligned} 2^{5/6}+2^{1/2}+2^{1/3}+1 &= 2^{5/6}+2^{3/6}+2^{2/6}+1 \\ &= 2^{3/6}\left(2^{2/6}+1\right)+2^{2/6}+1 \\ &= \left(2^{3/6}+1\right)\left(2^{2/6}+1\right) \\ &= \left(2^{1/2}+1\right)\left(2^{1/3}+1\right) \end{aligned}$$

The first factor can be rationalized using the difference of squares identity $(a+b)(a-b)=a^2-b^2$, with $a=2^{1/2}$ and $b=1$:

$$\left(2^{1/2}+1\right)\left(2^{1/2}-1\right)=\left(2^{1/2}\right)^2-1^2=1.$$

The second factor can be rationalized using the sum of cubes identity $(a+b)(a^2-ab+b^2)=a^3+b^3$, with $a=2^{1/3}$ and $b=1$:

$$\left(2^{1/3}+1\right)\left(\left(2^{1/3}\right)^2-2^{1/3}+1\right)=\left(2^{1/3}\right)^3+1^3=3.$$

Putting it all together:

$$\begin{aligned} \frac{1}{2^{5/6}+2^{1/2}+2^{1/3}+1} &= \frac{1}{\left(2^{1/2}+1\right)\left(2^{1/3}+1\right)}\frac{\left(2^{1/2}-1\right)\left(2^{2/3}-2^{1/3}+1\right)}{\left(2^{1/2}-1\right)\left(2^{2/3}-2^{1/3}+1\right)} \\ &= \frac{\left(2^{1/2}-1\right)\left(2^{2/3}-2^{1/3}+1\right)}{1\cdot 3} \\ &= \frac{1}{3}\left(2^{1/2}-1\right)\left(2^{2/3}-2^{1/3}+1\right). \end{aligned}$$

7. What is the value of the following series?

$$\sum_{k=0}^{2022} \frac{2022! \cdot (-1)^k \, 2^k}{k! \cdot (2022-k)!}$$

(a) $\dfrac{1-(-2)^{2023}}{3}$ **(b)** -1 **(c)** 1

(d) 3^{2022} **(e)** None of the above

SOLUTION **(c)**: Using the binomial Theorem,

$$\begin{aligned}\sum_{k=0}^{2022} \frac{2022!\,(-1)^k \, 2^k}{k! \cdot (2022-k)!} &= \sum_{k=0}^{2022} \frac{2022!}{k! \cdot (2022-k)!}\,(-2)^k \\ &= \sum_{k=0}^{2022} \binom{2022}{k} (-2)^k \, 1^{2022-k} \\ &= \sum_{k=0}^{2022} \binom{2022}{k} 1^{2022-k} \, (-2)^k \\ &= (1+(-2))^{2022} \\ &= (-1)^{2022} = 1\end{aligned}$$

8. Which of the following is equal to

$$\left(\sqrt[4]{27}+\sqrt{3}+\sqrt[4]{3}+1\right)^2?$$

(a) $\dfrac{4}{\sqrt{3}-2\sqrt[4]{3}+1}$ **(b)** $4\sqrt{3}+4$ **(c)** $2\sqrt[4]{27}+2\sqrt[4]{3}+6\sqrt{3}+6$

(d) $\dfrac{16}{\sqrt{3}+2\sqrt[4]{3}+1}$ **(e)** None of the above

SOLUTION **(a)**: Since

$$x^4-1=(x-1)\left(x^3+x^2+x+1\right)$$

(which can be derived by repeated use of the difference of squares identity followed by expanding $(x+1)(x^2+1)=x^3+x^2+x+1$),

$$2=3-1=\left(\sqrt[4]{3}\right)^4-1=\left(\sqrt[4]{3}-1\right)\left(\sqrt[4]{3}^3+\sqrt[4]{3}^2+\sqrt[4]{3}+1\right).$$

This gives us

$$2=\left(\sqrt[4]{3}-1\right)\left(\sqrt[4]{27}+\sqrt{3}+\sqrt[4]{3}+1\right).$$

Dividing both sides by $\left(\sqrt[4]{3}-1\right)$ and squaring both sides gives us

$$\left(\sqrt[4]{27}+\sqrt{3}+\sqrt[4]{3}+1\right)^2=\left(\frac{2}{\sqrt[4]{3}-1}\right)^2=\frac{4}{\sqrt{3}-2\sqrt[4]{3}+1}$$

9. Which of the following expressions are equivalent to

$$\frac{(3x+2)^{\frac{1}{2}}\left(\frac{1}{3}\right)(2x+3)^{-\frac{2}{3}}(2)-(2x+3)^{\frac{1}{3}}\left(\frac{1}{2}\right)(3x+2)^{-\frac{1}{2}}(3)}{\left[(3x+2)^{\frac{1}{2}}\right]^2}$$

(a) $\dfrac{-5}{6\,(2x+3)^{\frac{4}{3}}\,(3x+2)^2}$ **(b)** $\dfrac{2\sqrt{3x+2}-3\sqrt[3]{2x+3}}{6\sqrt{(3x+2)^3}\sqrt[3]{(2x+3)^2}}$ **(c)** 0

(d) $-\dfrac{6x+19}{6\,(2x+3)^{\frac{2}{3}}\,(3x+2)^{\frac{3}{2}}}$ **(e)** None of the above

SOLUTION **(d)**: Start by factoring the smallest power of each factor from the numerator:

$$\frac{(3x+2)^{-\frac{1}{2}}(2x+3)^{-\frac{2}{3}}\left[(3x+2)^{\frac{1}{2}-\left(-\frac{1}{2}\right)}\left(\frac{1}{3}\right)(2)-(2x+3)^{\frac{1}{3}-\left(-\frac{2}{3}\right)}\left(\frac{1}{2}\right)(3)\right]}{\left[(3x+2)^{\frac{1}{2}}\right]^2}$$

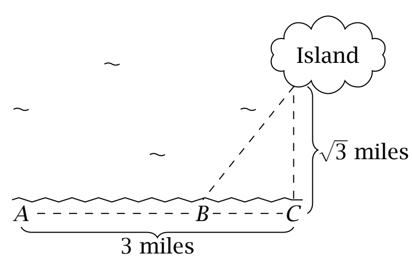
That can be further simplified:

$$\frac{(3x+2)^{-\frac{1}{2}}(2x+3)^{-\frac{2}{3}}\left[(3x+2)\left(\frac{2}{3}\right)-(2x+3)\left(\frac{3}{2}\right)\right]}{\left[(3x+2)^{\frac{1}{2}}\right]^2}.$$

Factoring $\frac{1}{6}$ from the numerator and moving factors with negative exponents to the denominator gives us

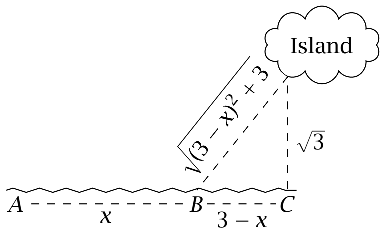
$$\frac{4(3x+2)-9(2x+3)}{6\left[(3x+2)^{\frac{1}{2}}\right]^3(2x+3)^{\frac{2}{3}}}=\frac{12x+8-18x-27}{6(3x+2)^{\frac{3}{2}}(2x+3)^{\frac{2}{3}}}=\frac{-6x-19}{6(3x+2)^{\frac{3}{2}}(2x+3)^{\frac{2}{3}}}=-\frac{6x+19}{6(3x+2)^{\frac{3}{2}}(2x+3)^{\frac{2}{3}}}$$

10. A person walks along a beach, starting at point A, at a rate of 3 mi/h and at point B, goes into the water and swims at a rate of 2 mi/h diagonally out to an island that is a distance of $\sqrt{3}$ mi from point C, directly across from the island on the shore, as shown in the picture. The total distance from point A to point C is 3 mi. There are two different choices for the distance, in miles, from point A to point B that will result in a total time for walking and swimming of one hour and 40 minutes; what is the sum of those numbers?



(a) 2 **(b)** 4 **(c)** $\frac{14}{5}$ **(d)** $\frac{16}{5}$ **(e)** None of the above

SOLUTION **(c)**: Let x be the distance from A to B. Then the distance from B to C is $3-x$, and from the Pythagorean Theorem, the distance from B to the island is $\sqrt{12-6x+x^2}$.



The total time from A to the island is then

$$\frac{x}{3}+\frac{\sqrt{12-6x+x^2}}{2}$$

This must be equal to 1 hour and 40 minutes, or $\frac{5}{3}$ of an hour.

$$\begin{aligned}
\frac{x}{3}+\frac{\sqrt{12-6x+x^2}}{2} &= \frac{5}{3}\\
2x+3\sqrt{12-6x+x^2} &= 10\\
3\sqrt{12-6x+x^2} &= 10-2x\\
9\left(12-6x+x^2\right) &= 100-40x+4x^2\\
108-54x+9x^2 &= 100-40x+4x^2\\
5x^2-14x+8 &= 0\\
(5x-4)(x-2) &= 0
\end{aligned}$$

and so the solutions are 2 and $\frac{4}{5}$.

$$2+\frac{4}{5}=\frac{14}{5}$$

olution]

11. Which of the following is the solution set to the inequality

$$\frac{1}{|x-3|} + \frac{2}{x-3} \le 5?$$

(a) $(-\infty, 3) \cup \left[\frac{18}{5}, \infty\right)$ **(b)** $\left(-\infty, \frac{16}{5}\right] \cup \left[\frac{18}{5}, \infty\right)$ **(c)** $(-\infty, 3] \cup \left[\frac{18}{5}, \infty\right)$

(d) $\left[\frac{16}{5}, \frac{18}{5}\right]$ **(e)** None of the above

SOLUTION **(a)**: We break this up into cases:

– Case $x > 3$, or $x - 3 > 0$: Then $|x-3| = x - 3$, and the inequality becomes

$$\frac{1}{x-3} + \frac{2}{x-3} \le 5$$

or

$$\frac{3}{x-3} \le 5$$

Since $x - 3 > 0$, multiplying by $x - 3$ gives us

$$\begin{aligned} 3 &\le 5(x-3) \\ 3 &\le 5x - 15 \\ 18 &\le 5x \\ x &\ge \frac{18}{5} \end{aligned}$$

– Case $x < 3$, or $x - 3 < 0$: Then $|x-3| = -(x-3)$, and the inequality becomes

$$-\frac{1}{x-3} + \frac{2}{x-3} \le 5$$

or

$$\frac{1}{x-3} \le 5$$

Since $x - 3 < 0$, multiplying by $x - 3$ gives us

$$\begin{aligned} 1 &\ge 5(x-3) \\ 1 &\ge 5x - 15 \\ 16 &\ge 5x \\ x &\le \frac{16}{5} \end{aligned}$$

Since $x < 3 < \frac{16}{5}$, this will be automatically true in this case.

Combining the two cases together will tell us that either $x < 3$ or $x \ge \frac{18}{5}$.
The soltution is the set $(-\infty, 3) \cup \left[\frac{18}{5}, \infty\right)$.

12. A game comes with a set of three fair six-sided dice. In one particular set, while two of the dice are regular, labeled with numbers 1 through 6, due to a manufacturing error, on the third die the side that is supposed to be labeled with 1 got labeled with 6 (so there are exactly two sides labeled 6 on the die).
Suppose two of the dice are chosen at random and rolled once. What is the probability that both land with a side labeled 6 facing up?

(a) $\frac{5}{108}$ **(b)** $\frac{1}{24}$ **(c)** $\frac{1}{18}$ **(d)** $\frac{1}{6}$ **(e)** None of the above

SOLUTION **(a)**: Define the following two events:

- Event R: selecting two regular dice (labeled 1 - 6).
- Event S: rolling two sixes.

Then

$$P(S) = P(R)\,P(S \mid R) + P(\overline{R})\,P(S \mid \overline{R})$$

There are several different ways to calculate $P(R)$:

- Supopse we label the three dice as 1,2 and 3, with 3 being the label of the mis-labeld die. Then there are 6 ways to select two dice: $(1,2)$, $(1,3)$, $(2,1)$, $(2,3)$, $(3,1)$ and $(3,2)$. Four of the six ways include the mis-labeled die, so there are 2 out of 6 ways to select the two regular dice. So $P(R) = 2/6 = 1/3$.
- Another way is to instead keep track of the die that is left out. There are three dice than can be left out, and one of them is the mis-labeled one, so in one of the three cases you end up selecting the two regular dice. Again, $P(R) = 1/3$.

For the conditional probability $P(S \mid R)$, we are rolling two dice, each with probability of rolling six equal to 1/6. The rolls are independent, so $P(S \mid R) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.
For the conditional probability $P(S \mid \overline{R})$, we are rolling two dice, one with probability of rolling six equal to 1/6, one with the probability of rolling six equal to 1/3. The rolls are independent, so $P(S \mid \overline{R}) = \frac{1}{6} \cdot \frac{1}{3} = \frac{1}{18}$.
So

$$\begin{aligned} P(S) &= \frac{1}{3}\frac{1}{36} + \frac{2}{3}\frac{1}{18} \\ &= \frac{1}{108} + \frac{1}{27} \\ &= \frac{1}{108} + \frac{4}{108} \\ &= \frac{5}{108} \end{aligned}$$

13. Evaluate $(1+i)^{2022}$.

(a) 1 **(b)** $-2^{1011}i$ **(c)** $2^{1011}i$

(d) $1011-2022i$ **(e)** None of the above

SOLUTION **(b)**: (Note that on the actual exam, there was a typo, so that both (b) and (c) were correct. When grading the exam, both of these answers were accepted.)
Since $1+i=\sqrt{2}e^{\pi i/4}$, $(1+i)^{2022}=2^{2022/2}e^{2022\pi i/4}=2^{1011}e^{1011\pi i/2}$. Further $e^{1011\pi i/2}=\left(e^{\pi i/2}\right)^{1011}=i^{1011}$.
Since $1011=1008+3=4\cdot 252+3$, $i^{1011}=i^3=-1$.
Putting all this together gives us

$$(1+i)^{2022}=2^{1011}(-i)=-2^{1011}i$$

14. Which of the following is equal to $\sin\left(\frac{\pi}{8}\right)$?

(a) $\sqrt{\dfrac{2-\sqrt{2}}{2}}$ **(b)** $\sqrt{\dfrac{2+\sqrt{2}}{2}}$ **(c)** $\dfrac{\sqrt{2-\sqrt{2}}}{2}$ **(d)** $\dfrac{\sqrt{2+\sqrt{2}}}{2}$ **(e)** None of the above

SOLUTION **(c)**: There are several ways one can proceed:

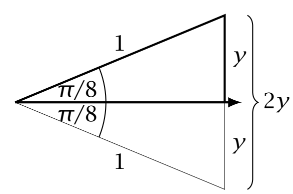
Using half angle formula The half angle formula for the sine function says

$$\sin\left(\frac{\theta}{2}\right) = (-1)^k\sqrt{\frac{1-\cos(\theta)}{2}}$$

where k is such that $k\pi \le \frac{\theta}{2} < (k+1)\pi$.
When $\frac{\theta}{2} = \frac{\pi}{8}$, we have $k = 0$ and $\theta = \frac{\pi}{4}$. Then

$$\begin{aligned}\sin\left(\frac{\pi}{8}\right) &= (-1)^0\sqrt{\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}} \\ &= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} \\ &= \sqrt{\frac{2-\sqrt{2}}{4}} \\ &= \frac{\sqrt{2-\sqrt{2}}}{2}\end{aligned}$$

Using the law of cosines In the triangle on the right, $\sin\left(\frac{\pi}{8}\right) = y$.
Applying the law of cosines, we get



$$(2y)^2 = 1^2 + 1^2 - 2\cdot 1\cdot 1\cdot\cos\left(\frac{\pi}{4}\right)$$

which will simplify to

$$4y^2 = 2-\sqrt{2}$$

Then $y^2 = \frac{2-\sqrt{2}}{4}$ and $y = \frac{\sqrt{2-\sqrt{2}}}{2}$.

Using complex numbers Let

$$z = \cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right) = e^{\frac{\pi i}{8}}$$

To shorten things up, lets denote the real part of z as c and the imaginary part as s. We want to find s.
We know that $z^2 = c^2 - s^2 + 2csi$. At the same time,

$$z^2 = e^{\frac{\pi i}{4}} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

Comparing the real and imaginary parts, we get

$$c^2 - s^2 = \frac{\sqrt{2}}{2}$$
$$2cs = \frac{\sqrt{2}}{2}$$

We are trying to solve for s. From the second equation, $c = \frac{\sqrt{2}}{4s}$. Substituting to the first equation will give us

$$\frac{1}{8s^2} - s^2 = \frac{\sqrt{2}}{2}$$
$$1 - 8s^4 = 4\sqrt{2}s^2$$
$$8s^4 + 4\sqrt{2}s^2 - 1 = 0$$

The last equation is a quadratic equation in s^2, and so using the quadratic formula

$$\begin{aligned} s^2 &= \frac{-4\sqrt{2} \pm \sqrt{\left(4\sqrt{2}\right)^2 - 4 \cdot 8 \cdot (-1)}}{2 \cdot 8} \\ &= \frac{-4\sqrt{2} \pm \sqrt{64}}{16} \\ &= \frac{-4\sqrt{2} + 8}{16} \\ &= \frac{-\sqrt{2} + 2}{4} \end{aligned}$$

Note that we droped the $-$ since s^2 cannot be negative.
Then, since $\sin\left(\frac{\pi}{4}\right)$ is positive, we get

$$s = \sqrt{\frac{2 - \sqrt{2}}{4}} = \frac{\sqrt{2 - \sqrt{2}}}{2}$$

15. Which of the following is a one-to-one function on the given domain?

(a) $\cos(x)$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ **(b)** $\sin(x)$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ **(c)** $x^3 - x$, $-1 \le x \le 1$

(d) $|x + 2|$, $-4 \le x \le 2$ **(e)** None of the above

Solution **(b)**:

- The function $\cos(x)$ is an even function, so cannot be one-to-one on any interval that contains 0 in its interior.
- The function $\sin(x)$ is one-to-one on any interval that corresponds to the left or to the right semicircle. Interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ corresponds to the right semicircle.
- The function $x^3 - x$ has zero at -1, 0 and 1, so it is not one-to-one on $[-1, 1]$,
- Finally, the function $|x + 2|$ has value 2 for $x = -4$ and $x = 0$, so it is not one-to-one on the given domain.

16. The matrix C is the product of two given matrices:

$$C = \begin{pmatrix} 1 & 3 & 2 \\ 5 & a & 1 \end{pmatrix} \cdot \begin{pmatrix} b & 2 & -3 \\ 1 & 2 & 5 \\ 2 & 3 & 1 \end{pmatrix}$$

If $c_{2,1} = -5$ and $c_{2,3} = 1$, what are a and b?

(a) $a = 3$ and $b = 2$ **(b)** $a = 2$ and $b = -2$ **(c)** $a = 5$ and $b = 1$

(d) $a = 3$ and $b = 1$ **(e)** None of the above

Solution **(e)**: If we call the first of the given matrices A and the second one B, then

$$\begin{aligned} c_{2,1} &= a_{2,1}b_{1,1} + a_{2,2}b_{2,1} + a_{2,3}b_{3,1} \\ &= 5 \cdot b + a \cdot 1 + 1 \cdot 2 \\ &= 5b + a + 2 \end{aligned}$$

and

$$\begin{aligned} c_{2,3} &= a_{2,1}b_{1,3} + a_{2,2}b_{2,3} + a_{2,3}b_{3,3} \\ &= 5 \cdot (-3) + a \cdot 5 + 1 \cdot 1 \\ &= -15 + 5a + 1 \\ &= 5a - 14 \end{aligned}$$

This gives us two equations

$$a + 5b + 2 = -5$$
$$5a - 14 = 1$$

From the second equation, $a = 3$. Substituting into the first equation gives $5b + 5 = -5$ or $b = -2$.

17. The function F, which takes functions as inputs and returns functions, is defined by

$$F(f)(x) = f(20x) + 22$$

If f is a linear function such that $f(5) = 16$ and $F(f)(5) = 323$, find the formula for $F(f)(x)$.

(a) $F(f)(x) = 2x + 8$

(b) $F(f)(x) = 20x^2 - 40x + 23$

(c) $F(f)(x) = 80x - 89$

(d) $F(f)(x) = 60x + 23$

(e) None of the above

SOLUTION **(d)**: Since f is linear, $f(x) = ax + b$ for some a and b. Then $F(f)(x) = f(20x) + 22 = a \cdot (20x) + b + 22$. Plugging in 5 into both f and $F(f)$ will give us two equations

$$5a + b = 16$$
$$100a + b + 22 = 323$$

This system of equations has a unique solution $a = 3$ and $b = 1$, so $f(x) = 3x + 1$, and $F(f)(x) = 60x + 23$.

18. The functions p and q are defined by the following table, in which some values are missing, which is indicated by a question mark. The table also contains some values of the function r, which is defined by the equation $r(x) = q(p(x))$. Again some values have been replaced by question marks.

x	**0**	**1**	**2**	**3**	**4**	**5**
$p(x)$	4	?	?	2	1	0
$q(x)$	?	2	?	0	5	1
$r(x)$	?	1	0	4	?	3

What is the sum of all the missing values in the table?

(a) 22 **(b)** 15 **(c)** 20 **(d)** 17

(e) Impossible to determine with given information.

SOLUTION **(a)**: First, we can find $r(0)$ and $r(4)$ by referring to the table:

$$r(0) = q(p(0)) = q(4) = 5 \text{ because } p(0) = 4.$$

Similarly,

$$r(4) = q(p(4)) = q(1) = 2 \text{ because } p(4) = 1.$$

Next, we find $p(1)$ and $p(2)$:

$$r(1) = q(p(1)) = 1 \text{ and since } q(5) = 1 \text{ we get } p(1) = 5.$$

Similarly,

$$r(2) = q(p(2)) = 0 \text{ gives us } p(2) = 3,$$

since $q(3) = 0$.
To find $q(0)$ and $q(2)$, we need to find for which x is $p(x) = 0$ or 5:

$$p(5) = 0 \text{ which means } q(0) = q(p(5)) = r(5) = 3,$$

and

$$p(3) = 2 \text{ which means } q(2) = q(p(3)) = r(3) = 4.$$

Now we need to add all the values:

$$r(0) + r(4) + p(1) + p(2) + q(0) + q(2) = 5 + 2 + 5 + 3 + 3 + 4 = 22.$$

19. The set $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ of all whole numbers has an associative operation of addition, denoted by $+$. It also contains a special object, 0, with the property that for any whole number n, $n + 0 = 0 + n = n$. We say that the set $\mathbb{N}$ is a *monoid* with operation $+$ and *identity object* 0. Another example of a monoid is the set $\mathbb{B} = \{0, 1\}$ with the operation $\vee$ (pronounced 'or') defined by the equations $0 \vee 0 = 0$, $0 \vee 1 = 1 \vee 0 = 1$ and $1 \vee 1 = 1$. The identity object in this monoid is also 0.
A function f from $\mathbb{N}_0$ to $\mathbb{B}$ is a *monoid morphism* if it satisfies the following two properties:

- For any two numbers m and n from the set $\mathbb{N}_0$, the equation $f(m + n) = f(m) \vee f(n)$ is true (we say that f preserves the monoid operations).
- $f(0) = 0$ (we say that f preserves the identity objects).

Is the function f defined by

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

a monoid morphism between the two examples of monoids described above?

(a) f is a monoid morphism.

(b) f preserves the monoid operation but not the identity objects.

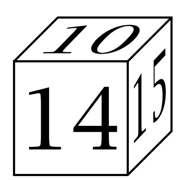
(c) f preserves the identity objects but not the monoid operation.

(d) f preserves neither the identity objects nor the monoid operation.

(e) f is not a function from $\mathbb{N}_0$ to $\mathbb{B}$.

Solution **(c)**: First f is a function from $\mathbb{N}_0$ to $\mathbb{B}$: since every element of $\mathbb{N}_0$ is either even or odd, f assigns to it a unique element of $\mathbb{B}$.
Since the identity object in $\mathbb{N}_0$ is 0, which is even, $f(0) = 0$, which is the identity element in $\mathbb{B}$. Therefore f preserved the identity objects.
However, since $f(1 + 1) = f(2) = 0$, while $f(1) \vee f(1) = 1 \vee 1 = 1$, we see that $f(1 + 1) \neq f(1) \vee f(1)$, so the function f does not preserve the monoid operation.

20. Each side of the cube depicted on the right is numbered with a positive integer in such a way that the products of the numbers on each pair of opposite sides are all the same. Find the lowest possible sum of all the numbers on the sides of the cube.



(a) 78 **(b)** 80 **(c)** 89 **(d)** 107 **(e)** None of the above

Solution **(c)**: Since the products of the numbers on the opposite sides must be the same, they must be a multiple of 10, 14 and 15. The get the lowest possible sum, we must use the least common multiple. Since $10 = 2 \cdot 5$, $14 = 2 \cdot 7$ and $15 = 3 \cdot 5$, we need $2 \cdot 3 \cdot 5 \cdot 7 = 210$. So the number opposite to 10 is 21, opposite to 14 is 15, and opposite to 15 is 14. The sum of all the numbers is $10 + 21 + 14 + 15 + 15 + 14 = 89$.

21. Find the largest integer smaller than $\sqrt{22+\sqrt{22+\sqrt{22+\sqrt{22}}}}$.

(a) 4 **(b)** 5 **(c)** 6 **(d)** 9 **(e)** 22

SOLUTION **(b)**: First, $\sqrt{22+\sqrt{22+\sqrt{22+\sqrt{22}}}} > \sqrt{22+\sqrt{22}} > \sqrt{22+4} > 5$.
Then, if we define a recursive sequence $a_0 = \sqrt{22}$, $a_{n+1} = \sqrt{22+a_n}$ for $n \geq 0$, our number is a_3. Clearly $a_0 < 6$. If $a_n < 6$, then $a_{n+1} = \sqrt{22+a_n} < \sqrt{22+6} < 6$. That means $a_n < 6$ for all $n \geq 0$. So our number is greater than 5, but less than 6.

22. How many pairs of integers (x, y) are solutions of the equation

$$3x^2y - 10xy - 8y - 17 = 0?$$

(a) none **(b)** one **(c)** two **(d)** four **(e)** None of the above

SOLUTION **(b)**: We can rewrite the equation as

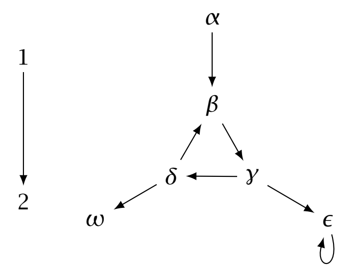
$$(3x^2 - 10x - 8)y = 17$$

If x is an integer, so is $3x^2 - 10x - 8$. There are only four ways to write 17 as a product of two integers: $17 \cdot 1$, $1 \cdot 17$, $(-17) \cdot (-1)$ and $(-1) \cdot (-17)$, which gives us four possibilities for $3x^2 - 10x - 8$:

- $3x^2 - 10x - 8 = 17$, or $3x^2 - 10x - 25 = 0$. The left side factors: $(3x + 5)(x - 5) = 0$, so we have an integer solution $x = 5$.
- $3x^2 - 10x - 8 = 1$, or $3x^2 - 10x - 9 = 0$. The left side does not factor using integers, there are no integer solutions.
- $3x^2 - 10x - 8 = -17$, or $3x^2 - 10x + 9 = 0$. The left side does not factor using integers, there are no integer solutions.
- $3x^2 - 10x - 8 = -1$, or $3x^2 - 10x - 7 = 0$. The left side does not factor using integers, there are no integer solutions.

Therefore the only pair of integer solutions is $(5, 1)$.

23. A diagram consists of a set of nodes and a set of arrows between nodes. Two diagrams are shown on the right, first with the set of nodes $A = \{1, 2\}$, the second with the set of nodes $B = \{\alpha, \beta, \gamma, \delta, \epsilon, \omega\}$. In the first diagram, you can get from 1 to 2 by following arrows, but not from 2 to 1. In the second diagram, you can for example get from α to every node *except* α by following arrows. Two other extremes in the second diagram are ϵ, from which you can only get to itself, and ω from which you cannot get anywhere by following arrows.



A function f from the set A to the set B is called a *diagram mapping* if for every two nodes a and b from the set A such that you can get from a to b by following arrows, you can also get from $f(a)$ to $f(b)$ by following arrows.

How many diagram mappings are there from the first diagram to the second diagram?

(a) 6 **(b)** 17 **(c)** 18 **(d)** 20 **(e)** 21

SOLUTION **(e)**: Each function from the set $A = \{1, 2\}$ can be uniquely defined by specifying $f(1)$ and $f(2)$. Let's divide the problem into cases:

- Suppose $f(1) = \alpha$. Since we can get from α to any other node of the second diagram, $f(2)$ can be any node of the second diagram except α. So $f(2) = \beta$, $f(2) = \gamma$, $f(2) = \delta$, $f(2) = \epsilon$ and $f(2) = \omega$ are all valid choices if we want f to be a diagram mapping. Therefore there are 5 diagram mapping from the first diagram to the second one that map 1 to α.

- Suppose $f(1) = \beta$. Since we can get from β to any node of the second diagram except α (including β itself), $f(2)$ can be any node of the second diagram except α. So $f(2) = \beta$, $f(2) = \gamma$, $f(2) = \delta$, $f(2) = \epsilon$ and $f(2) = \omega$ are all valid choices if we want f to be a diagram mapping. Again, there are 5 diagram mapping from the first diagram to the second one that map 1 to β.

- Suppose $f(1) = \gamma$. Again, we can get from γ to any node of the second diagram except α (including γ itself), so there are 5 diagram mapping from the first diagram to the second one that map 1 to γ.

- Suppose $f(1) = \delta$. The same reasoning can be applied to see that there are 5 diagram mapping from the first diagram to the second one that map 1 to δ.

- Now suppose $f(1) = \epsilon$. There is only one node in the second diagram that can be accessed from ϵ by following arrows, and that is ϵ itself. Therefore if $f(1) = \epsilon$, $f(2)$ must also be ϵ, and there is only one diagram mapping such that $f(1) = \epsilon$.

- Finally, suppose $f(1) = \omega$. There is no node in the second diagram accessible from ω by folowing arrows, se we do not have any node to map 2 to. There is no diagram mapping that would map 1 to ω.

Therefore the total number of diagram mappings from the first diagram to the second diagram is $5 + 5 + 5 + 5 + 1 + 0 = 21$.

24. Assuming that 'wigglers' are those who wiggle, 'wobblers' are those that wobble, and 'wagglers' are those who waggle, which of the following sets of premises will necessarily lead to the conclusion that "Wilbur is not a weeble"?

(a) All weebles wobble.
No wobblers wiggle.
Some wigglers waggle.
Wilbur waggles.

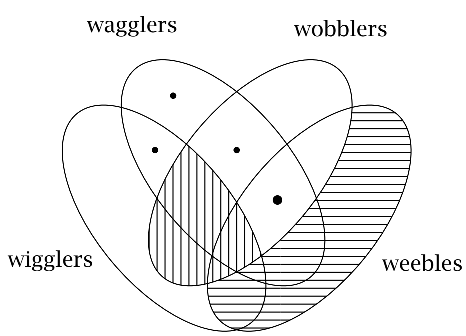
(b) Some weebles wobble.
No wobblers wiggle.
Some wagglers wiggle.
Wilbur waggles.

(c) All weebles wobble.
No wobblers wiggle.
All wigglers waggle.
Wilbur waggles.

(d) All weebles wobble.
No wobblers wiggle.
All wagglers wiggle.
Wilbur waggles.

(e) None of the above

SOLUTION **(d)**: These are all arguments involving four sets: wigglers, wagglers, wobblers and weebles. We can visualize the premises using Venn diagrams with four sets. For each premise, we will shade the area that is guaranteed empty by that premise. Possible locations of Wilbur will be indicated by a dot.
First set of premises:



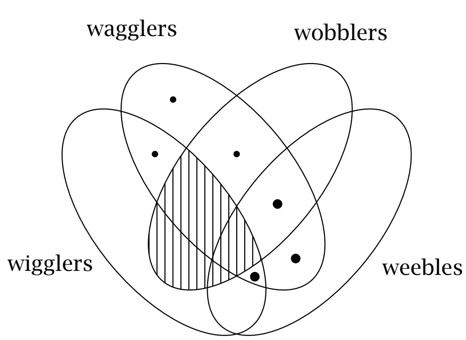
– "All weebles wobble." Shade the area of weebles that is ouside of wobblers with

– "No wobblers wiggle." Shade the area of wobblers that is inside of wigglers with

– "Some wigglers waggle." This just guarantees that certain areas of the diagram will be non-empty, and has no relevance for Wilbur, as long as it is consistent with the other premises.

– "Wilbur waggles." There are four non-shaded areas inside the wagglers where Wilbur can be, as indicated by the dots.

We can see that Wilbur can be a weeble.
Second set of premises:



– "Some weebles wobble." This just guarantees that certain areas of the diagram will be non-empty, and has no relevance for Wilbur, as long as it is consistent with the other premises.

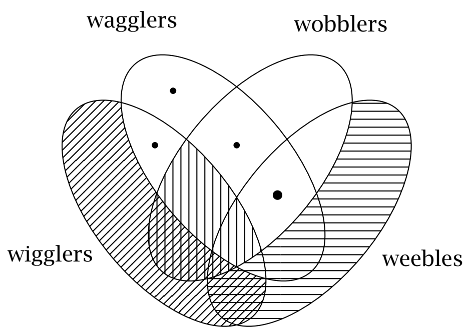
– "No wobblers wiggle." Shade the area of wobblers that is inside of wigglers with

– "Some wigglers waggle." This just guarantees that certain areas of the diagram will be non-empty, and has no relevance for Wilbur, as long as it is consistent with the other premises.

- "Wilbur waggles." There are six non-shaded areas inside the wagglers where Wilbur can be, as indicated by the dots.

We can see that Wilbur can be a weeble.

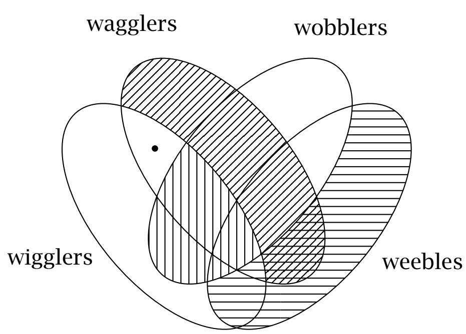
Third set of premises:



- "All weebles wobble." Shade the area of weebles that is ouside of wobblers with
- "No wobblers wiggle." Shade the area of wobblers that is inside of wigglers with
- "All wigglers waggle." Shade the area of wigglers that is ouside of wagglers with
- "Wilbur waggles." There are four non-shaded areas inside the wagglers where Wilbur can be, as indicated by the dots. Some of these dots are inside weebles.

We can see that Wilbur can be a weeble.

Fourth set of premises:



- "All weebles wobble." Shade the area of weebles that is ouside of wobblers with
- "No wobblers wiggle." Shade the area of wobblers that is inside of wigglers with
- "All wagglers wiggle." Shade the area of wagglers that is ouside of wigglers with
- "Wilbur waggles." There is only one non-shaded area inside the wagglers where Wilbur can be, as indicated by the dot.

We can see that with this set of premises, Wilbur can not be a weeble.

25. Suppose that x and y satisfy $\frac{x-y}{x+y} = 9$ and $\frac{xy}{x+y} = -60$. The value of $(x+y)+(x-y)+xy$ is

(a) -50 **(b)** -150 **(c)** -14310 **(d)** 210 **(e)** 14160

SOLUTION **(b)**: Let $V = (x+y)+(x-y)+xy$. Then

$$\frac{V}{x+y} = \frac{x+y}{x+y} + \frac{x-y}{x+y} + \frac{xy}{x+y} = 1 + 9 - 60 = -50$$

and so $V = -50(x+y)$. Let $n = x+y$. Then $x-y = 9(x+y) = 9n$. Squaring both sides gives us $x^2 - 2xy + y^2 = 81n^2$. Since $x+y = n$, we also get $x^2 + 2xy + y^2 = n^2$. Subtracting the two equations will give us $-4xy = 80n^2$ or $xy = -20n^2$.
At the same time we know that $\frac{xy}{n} = -60$. So

$$-60 = \frac{xy}{n} = \frac{-20n^2}{n} = -20n$$

which means that $n = 3$.
Then $V = -50n = -150$.