Saginaw Valley State University

2022 Math Olympics — Level II

1. For $n \geq 2$, let k_n be the product $5 \cdot 10 \cdot 17 \cdot 26 \cdot \cdots \cdot (n^2+1)$, and let

$$S_n = \left(1 - \frac{1}{2^4}\right)\left(1 - \frac{1}{3^4}\right)\cdots\left(1 - \frac{1}{n^4}\right).$$

What is S_n/k_n?

(a) $\frac{1}{2}\frac{1}{(n!)^2}\left(1+\frac{1}{n}\right)$ **(b)** $\frac{1}{k_n}\frac{n^2}{n^4+1}$ **(c)** $\frac{n^4+1}{2n+2}\cdot\frac{1}{n!}$

(d) $\frac{1}{k_n}\frac{1}{n^4}\frac{1}{((n-1)!)^4}$ **(e)** None of the above

2. Consider the following equation:

$$3^x - 6^x = 12^x - \left(\frac{3}{2}\right)^x.$$

Which of the following is NOT a solution?

(a) $\frac{3\pi i}{\ln 2}$ **(b)** 0 **(c)** $\frac{6\pi i}{\ln 2}$ **(d)** $\frac{27\pi i}{\ln 2}$

(e) All are solutions

3. Consider the equation $p(x) : ax^2 + bx + c = 0$ whose coefficients a, b and c are all non-zero, and each of them satisfies an equation that results from removing the term containing that coefficent from the equation $p(x)$; for example, the coefficient b is a solution of the equation $ax^2 + c = 0$. What is the sum of all solutions of $p(x)$?

(a) Always 1 **(b)** Always -1 **(c)** Always 2 **(d)** 1 or -1 **(e)** 1 or 2

4. Find the sum of the first five terms of the series

$$\left\{\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots\right\}.$$

(a) $\frac{64}{81}$ **(b)** $\frac{550}{243}$ **(c)** $\frac{32}{81}$ **(d)** $\frac{422}{243}$ **(e)** None of the above

5. Suppose f is a function such that $f(n+1) = f(n) + 3f(n-1)$, for all integers n. If $f(4) = 151$ and $f(1) = 1$, what is $f(-1)$?

(a) 12 **(b)** 4 **(c)** $-\frac{11}{3}$ **(d)** $\frac{13}{3}$ **(e)** None of the above

6. Rationalize the denominator:

$$\frac{1}{2^{5/6} + 2^{1/2} + 2^{1/3} + 1}$$

(a) $\frac{1}{6}\left(2^{1/2} + 2^{1/3} - 2^{1/6}\right)$ **(b)** $\frac{1}{3}\left(2^{1/2} - 1\right)\left(2^{2/3} - 2^{1/3} + 1\right)$

(c) $\frac{1}{2}\left(2^{7/6} - 2^{5/6} + 1\right)$ **(d)** $\left(2^{1/2} - 1\right)\left(2^{2/3} - 2^{1/3} + 1\right)$

(e) None of the above

7. What is the value of the following series?

$$\sum_{k=0}^{2022} \frac{2022! \cdot (-1)^k\, 2^k}{k! \cdot (2022 - k)!}$$

(a) $\frac{1 - (-2)^{2023}}{3}$ **(b)** -1 **(c)** 1

(d) 3^{2022} **(e)** None of the above

8. Which of the following is equal to

$$\left(\sqrt[4]{27} + \sqrt{3} + \sqrt[4]{3} + 1\right)^2?$$

(a) $\frac{4}{\sqrt{3} - 2\sqrt[4]{3} + 1}$ **(b)** $4\sqrt{3} + 4$ **(c)** $2\sqrt[4]{27} + 2\sqrt[4]{3} + 6\sqrt{3} + 6$

(d) $\frac{16}{\sqrt{3} + 2\sqrt[4]{3} + 1}$ **(e)** None of the above

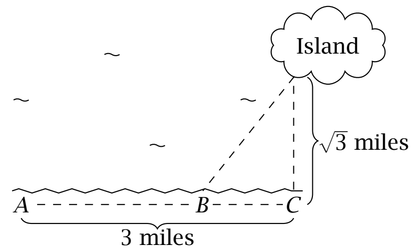
9. Which of the following expressions are equivalent to

$$\frac{(3x+2)^{\frac{1}{2}}\left(\frac{1}{3}\right)(2x+3)^{-\frac{2}{3}}(2) - (2x+3)^{\frac{1}{3}}\left(\frac{1}{2}\right)(3x+2)^{-\frac{1}{2}}(3)}{\left[(3x+2)^{\frac{1}{2}}\right]^2}$$

(a) $\frac{-5}{6\,(2x+3)^{\frac{4}{3}}\,(3x+2)^2}$ **(b)** $\frac{2\sqrt{3x+2} - 3\sqrt[3]{2x+3}}{6\sqrt{(3x+2)^3}\sqrt[3]{(2x+3)^2}}$ **(c)** 0

(d) $-\frac{6x+19}{6\,(2x+3)^{\frac{2}{3}}\,(3x+2)^{\frac{3}{2}}}$ **(e)** None of the above

10. A person walks along a beach, starting at point A, at a rate of 3 mi/h and at point B, goes into the water and swims at a rate of 2 mi/h diagonally out to an island that is a distance of $\sqrt{3}$ mi from point C, directly across from the island on the shore, as shown in the picture. The total distance from point A to point C is 3 mi. There are two different choices for the distance, in miles, from point A to point B that will result in a total time for walking and swimming of one hour and 40 minutes; what is the sum of those numbers?



(a) 2 **(b)** 4 **(c)** $\frac{14}{5}$ **(d)** $\frac{16}{5}$ **(e)** None of the above

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11. Which of the following is the solution set to the inequality

$$\frac{1}{|x-3|} + \frac{2}{x-3} \leq 5?$$

(a) $(-\infty, 3) \cup \left[\frac{18}{5}, \infty\right)$ **(b)** $\left(-\infty, \frac{16}{5}\right] \cup \left[\frac{18}{5}, \infty\right)$ **(c)** $(-\infty, 3] \cup \left[\frac{18}{5}, \infty\right)$

(d) $\left[\frac{16}{5}, \frac{18}{5}\right]$ **(e)** None of the above

12. A game comes with a set of three fair six-sided dice. In one particular set, while two of the dice are regular, labeled with numbers 1 through 6, due to a manufacturing error, on the third die the side that is supposed to be labeled with 1 got labeled with 6 (so there are exactly two sides labeled 6 on the die).
Suppose two of the dice are chosen at random and rolled once. What is the probability that both land with a side labeled 6 facing up?

(a) $\frac{5}{108}$ **(b)** $\frac{1}{24}$ **(c)** $\frac{1}{18}$ **(d)** $\frac{1}{6}$ **(e)** None of the above

13. Evaluate $(1+i)^{2022}$.

(a) 1 **(b)** $-2^{1011}i$ **(c)** $2^{1011}i$

(d) $1011 - 2022i$ **(e)** None of the above

14. Which of the following is equal to $\sin\left(\frac{\pi}{8}\right)$?

(a) $\sqrt{\frac{2-\sqrt{2}}{2}}$ **(b)** $\sqrt{\frac{2+\sqrt{2}}{2}}$ **(c)** $\frac{\sqrt{2-\sqrt{2}}}{2}$ **(d)** $\frac{\sqrt{2+\sqrt{2}}}{2}$ **(e)** None of the above

15. Which of the following is a one-to-one function on the given domain?

(a) $\cos(x)$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

(b) $\sin(x)$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ **(c)** $x^3 - x$, $-1 \leq x \leq 1$

(d) $|x+2|$, $-4 \leq x \leq 2$ **(e)** None of the above

16. The matrix C is the product of two given matrices:

$$C = \begin{pmatrix} 1 & 3 & 2 \\ 5 & a & 1 \end{pmatrix} \cdot \begin{pmatrix} b & 2 & -3 \\ 1 & 2 & 5 \\ 2 & 3 & 1 \end{pmatrix}$$

If $c_{2,1} = -5$ and $c_{2,3} = 1$, what are a and b?

(a) $a = 3$ and $b = 2$ **(b)** $a = 2$ and $b = -2$ **(c)** $a = 5$ and $b = 1$

(d) $a = 3$ and $b = 1$ **(e)** None of the above

17. The function F, which takes functions as inputs and returns functions, is defined by

$$F(f)(x) = f(20x) + 22$$

If f is a linear function such that $f(5) = 16$ and $F(f)(5) = 323$, find the formula for $F(f)(x)$.

(a) $F(f)(x) = 2x + 8$ **(b)** $F(f)(x) = 20x^2 - 40x + 23$

(c) $F(f)(x) = 80x - 89$ **(d)** $F(f)(x) = 60x + 23$

(e) None of the above

18. The functions p and q are defined by the following table, in which some values are missing, which is indicated by a question mark. The table also contains some values of the function r, which is defined by the equation $r(x) = q(p(x))$. Again some values have been replaced by question marks.

x	**0**	**1**	**2**	**3**	**4**	**5**
$p(x)$	4	?	?	2	1	0
$q(x)$	?	2	?	0	5	1
$r(x)$	?	1	0	4	?	3

What is the sum of all the missing values in the table?

(a) 22 **(b)** 15 **(c)** 20 **(d)** 17

(e) Impossible to determine with given information.

19. The set $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ of all whole numbers has an associative operation of addition, denoted by $+$. It also contains a special object, 0, with the property that for any whole number n, $n + 0 = 0 + n = n$. We say that the set $\mathbb{N}$ is a *monoid* with operation $+$ and *identity object* 0. Another example of a monoid is the set $\mathbb{B} = \{0, 1\}$ with the operation $\vee$ (pronounced 'or') defined by the equations $0 \vee 0 = 0$, $0 \vee 1 = 1 \vee 0 = 1$ and $1 \vee 1 = 1$. The identity object in this monoid is also 0.
    A function f from $\mathbb{N}_0$ to $\mathbb{B}$ is a *monoid morphism* if it satisfies the following two properties:

    – For any two numbers m and n from the set $\mathbb{N}_0$, the equation $f(m + n) = f(m) \vee f(n)$ is true (we say that f preserves the monoid operations).

    – $f(0) = 0$ (we say that f preserves the identity objects).

    Is the function f defined by

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

    a monoid morphism between the two examples of monoids described above?

    (a) f is a monoid morphism.

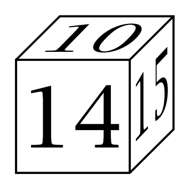
    (b) f preserves the monoid operation but not the identity objects.

    (c) f preserves the identity objects but not the monoid operation.

    (d) f preserves neither the identity objects nor the monoid operation.

    (e) f is not a function from $\mathbb{N}_0$ to $\mathbb{B}$.

20. Each side of the cube depicted on the right is numbered with a positive integer in such a way that the products of the numbers on each pair of opposite sides are all the same. Find the lowest possible sum of all the numbers on the sides of the cube.



    (a) 78 **(b)** 80 **(c)** 89 **(d)** 107 **(e)** None of the above

21. Find the largest integer smaller than $\sqrt{22 + \sqrt{22 + \sqrt{22 + \sqrt{22}}}}$.

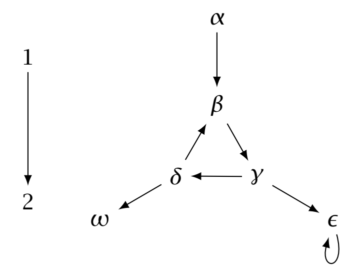
    (a) 4 **(b)** 5 **(c)** 6 **(d)** 9 **(e)** 22

22. How many pairs of integers (x, y) are solutions of the equation

$$3x^2y - 10xy - 8y - 17 = 0?$$

    (a) none **(b)** one **(c)** two **(d)** four **(e)** None of the above

23. A diagram consists of a set of nodes and a set of arrows between nodes. Two diagrams are shown on the right, first with the set of nodes $A = \{1, 2\}$, the second with the set of nodes $B = \{\alpha, \beta, \gamma, \delta, \epsilon, \omega\}$. In the first diagram, you can get from 1 to 2 by following arrows, but not from 2 to 1. In the second diagram, you can for example get from α to every node *except* α by following arrows. Two other extremes in the second diagram are ϵ, from which you can only get to itself, and ω from which you cannot get anywhere by following arrows.



A function f from the set A to the set B is called a *diagram mapping* if for every two nodes a and b from the set A such that you can get from a to b by following arrows, you can also get from $f(a)$ to $f(b)$ by following arrows.

How many diagram mappings are there from the first diagram to the second diagram?

(a) 6 **(b)** 17 **(c)** 18 **(d)** 20 **(e)** 21

24. Assuming that 'wigglers' are those who wiggle, 'wobblers' are those that wobble, and 'wagglers' are those who waggle, which of the following sets of premises will necessarily lead to the conclusion that "Wilbur is not a weeble"?

(a) All weebles wobble.
No wobblers wiggle.
Some wigglers waggle.
Wilbur waggles.

(b) Some weebles wobble.
No wobblers wiggle.
Some wagglers wiggle.
Wilbur waggles.

(c) All weebles wobble.
No wobblers wiggle.
All wigglers waggle.
Wilbur waggles.

(d) All weebles wobble.
No wobblers wiggle.
All wagglers wiggle.
Wilbur waggles.

(e) None of the above

25. Suppose that x and y satisfy $\frac{x-y}{x+y} = 9$ and $\frac{xy}{x+y} = -60$. The value of $(x + y) + (x - y) + xy$ is

(a) -50 **(b)** -150 **(c)** -14310 **(d)** 210 **(e)** 14160