Saginaw Valley State University
2022 Math Olympics — Level I Solutions

1. Consider the equation $p(x): ax^2 + bx + c = 0$ whose coefficients a, b and c are all non-zero, and each of them satisfies an equation that results from removing the term containing that coefficent from the equation $p(x)$; for example, the coefficient b is a solution of the equation $ax^2 + c = 0$. What is the sum of all solutions of $p(x)$?

   (a) Always 1 **(b)** Always -1 **(c)** Always 2 **(d)** 1 or -1 **(e)** 1 or 2

   SOLUTION **(d)**: This is what we know about the coefficients a, b and c:

   - a is a solution of $bx + c = 0$, so $ab + c = 0$.
   - b is a solution of $ax^2 + c = 0$, so $ab^2 + c = 0$.
   - c is a solution of $ax^2 + bx = 0$, so $ac^2 + bc = 0$.
   - a, b and c are all non-zero.

   From the first two equalities we get $ab = ab^2$, and since a and b are non-zero, we get $b = 1$.

   - With $b = 1$, the first two equations both become $a + c = 0$
   - The third equation becomes $ac^2 + c = 0$.

   From these two equalities we get $ac^2 = a$ and since $a \neq 0$, we get $c^2 = 1$ or $c = \pm 1$. Since $a + c = 0$, $a = -c = \mp 1$.
   That gives us only two possibilities for $p(x)$:

   - One is $p(x): x^2 + x - 1 = 0$. The solutions are
   $$\frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$
   The sum of these two solutions is -1.
   - The other is $p(x): -x^2 + x + 1 = 0$. Then the solutions are
   $$\frac{-1 \pm \sqrt{1+4}}{-2} = \frac{-1 \pm \sqrt{5}}{-2}$$
   The sum of these two solutions is 1.

   Alternatively, instead of discussing the two possibilities for $p(x)$, we can rewrite the equation as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. Since $\frac{c}{a} = -1 < 0$, the equation must have two real solutions. The sum of the two solutions is $-\frac{b}{a} = \pm 1$.

2. A marching band has 150 members. One day only part of them show up, but nobody wants to take the time to count how many there are. They first line up in rows with 5 members each, but there is one left over. Then they try rows with 6 members each, but there is still one left over. Then they try rows of 7, but there are two left over. How many people should line up in each row so no members will be left over?

(a) 4 **(b)** 11 **(c)** 13 **(d)** 17 **(e)** None of the above

SOLUTION **(b)**: There is 1 person left over with both rows of 5 and 6. If we removed this one person, the resulting number would be a multiple of both 5 and 6, and therefore a multiple of 30. Therefore the original number must be one more than a multiple of 30.
At the same time, there cannot be more than 150 people, so the possible numbers are 1, 31, 61, 91 and 121.
The number also must give the remainder of 2 when divided by 7. Out of the 5 candidate numbers, only 121 saisfies this condition.
Out of the suggested solutions, 11 is the only one that divides 121.

3. Which one of the following is the only true statement?

(a) The graph of a horizontal line can't have any x-intercepts.

(b) The graph of a horizontal line can't have a unique x-intercept, but may have more than one x-intercept.

(c) The graph of a parabola can't have a unique x-intercept, but may have more than one x-intercept.

(d) The graph of a polynomial of degree three or higher must have at least one x-intercept.

(e) Either none are true or more than one are true.

SOLUTION **(b)**: The graph of a horizontal line has no x-intercepts if it is the graph of the equation $y = b$, where b is a nonzero real number, but it has infinitely many x-intercepts if it is the line $y = 0$.
The graph of a parabola will have a unique x-intercept if the vertex is on the x-axis (in other words, if th equadratic function we are graphing is a perfect square).
Graph of a polynomial of degree three or higher can have no x-intercepts if it is of even degree, for example $y = x^4 + 1$.

4. Which of the following is equal to 10!? (If you are not familiar with the notation, the number $n!$ is defined for any non-negative integer using the following recursive formula:

$$0! = 1$$
$$n! = n \cdot ((n-1)!) \text{ for } n > 0)$$

(a) $5! \cdot 2!$ **(b)** $7! \cdot 5! \cdot 3!$ **(c)** $7! \cdot 5! \cdot 2!$ **(d)** $7! \cdot 5! \cdot 3! \cdot 2!$

(e) None of the above or more than one of the above.

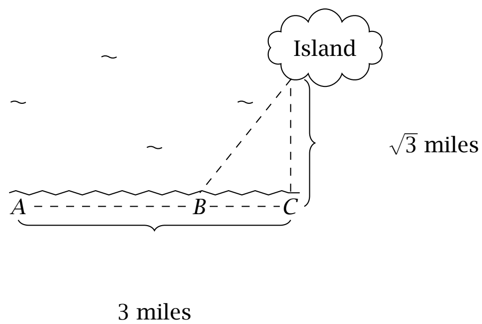
SOLUTION **(b)**: $10! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10$. $7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \equiv (2 \cdot 4) \cdot 3 \cdot 5 = 8 \cdot 3 \cdot 5$, and $3! = 1 \cdot 2 \cdot 3$, so $7! \cdot 5! \cdot 3! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot (3 \cdot 3) \cdot (5 \cdot 2) = 10!$.
To see that the others don't work: $5! \cdot 2!$ will not have a factor of 7, $7! \cdot 5! \cdot 2!$ does not have another factor of 3 to give a factor of 9, and $7! \cdot 5! \cdot 3! \cdot 2!$ has too many factors of 2, making it equal to $2(10!)$.

5. The base 6 expansion of one number is 550. The base 5 expansion of another number is 3440. What is their greatest common divisor expanded in base 4?

(a) 24 **(b)** 33 **(c)** 113 **(d)** 223 **(e)** None of the above

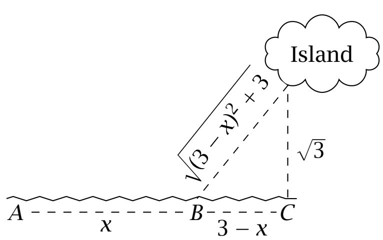
SOLUTION **(b)**: The first number is $550_6 = 5\times6^2+5\times6+0 = 180_{10}+30_{10} = 210_{10} = 2\times3\times5\times7$. The second number is $3440_5 = 3\times5^3+4\times5^2+4\times5+2 = 375_{10}+100_{10}+20_{10} = 495_{10} = 3\times3\times5\times11_{10}$. So their greatest common divisor is $3\times5 = 15_{10} = 3\times4+3 = 33_4$.

6. A person walks along a beach, starting at point A, at a rate of 3 mi/h and at point B, goes into the water and swims at a rate of 2 mi/h diagonally out to an island that is a distance of $\sqrt{3}$ mi from point C, directly across from the island on the shore, as shown in the picture. The total distance from point A to point C is 3 mi. There are two different choices for the distance, in miles, from point A to point B that will result in a total time for walking and swimming of one hour and 40 minutes; what is the sum of those numbers?



(a) 2 **(b)** 4 **(c)** $\frac{14}{5}$ **(d)** $\frac{16}{5}$ **(e)** None of the above

SOLUTION **(c)**: Let x be the distance from A to B. Then the distance from B to C is $3-x$, and from the Pythagorean Theorem, the distance from B to the island is $\sqrt{12-6x+x^2}$.



The total time from A to the island is then

$$\frac{x}{3} + \frac{\sqrt{12-6x+x^2}}{2}$$

This must be equal to 1 hour and 40 minutes, or $\frac{5}{3}$ of an hour.

$$\begin{aligned}
\frac{x}{3} + \frac{\sqrt{12-6x+x^2}}{2} &= \frac{5}{3} \\
2x + 3\sqrt{12-6x+x^2} &= 10 \\
3\sqrt{12-6x+x^2} &= 10-2x \\
9\left(12-6x+x^2\right) &= 100-40x+4x^2 \\
108-54x+9x^2 &= 100-40x+4x^2 \\
5x^2-14x+8 &= 0 \\
(5x-4)(x-2) &= 0
\end{aligned}$$

and so the solutions are 2 and $\frac{4}{5}$.

$$2 + \frac{4}{5} = \frac{14}{5}$$

7. Which of the following is equal to $|\sqrt{2022} - 45|$?

   (a) $\sqrt{3}$ **(b)** $\sqrt{1977}$ **(c)** $\sqrt{2022} - 45$ **(d)** $\sqrt{2022} + 45$

   (e) $45 - \sqrt{2022}$

   SOLUTION **(e)**: Since $45^2 = (40+5)^2 = 1600 + 2 \cdot 40 \cdot 5 + 25 = 2025 > 2022$, we know that $\sqrt{2022} - 45 < 0$ and so $|\sqrt{2022} - 45| = -(\sqrt{2022} - 45) = 45 - \sqrt{2022}$.

8. Which of the following is equal to $\sqrt[6]{s^5}\sqrt[9]{s}$ for all non-negative values of s?

   (a) $\sqrt[3]{s}$ **(b)** $\sqrt[5]{s^2}$ **(c)** $\sqrt[54]{s^5}$ **(d)** $\sqrt[18]{s^{17}}$ **(e)** None of the above

   SOLUTION **(d)**: Rewriting the radicals using fractional exponents, we get

$$x^{\frac{5}{6}}x^{\frac{1}{9}} = x^{\frac{5}{6}+\frac{1}{9}}.$$

   Adding the fractions will give us

$$\frac{5}{6} + \frac{1}{9} = \frac{15}{18} + \frac{2}{18} = \frac{17}{18}.$$

   Therefore

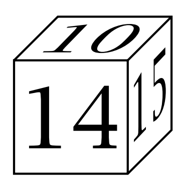
$$x^{\frac{5}{6}}x^{\frac{1}{9}} = x^{\frac{17}{18}} = \sqrt[18]{x^{17}}.$$

9. If 7^3 cubes are stacked to form a $7 \times 7 \times 7$ cube, how many small cubes are on the surface of the large cube?

   (a) 127 **(b)** 134 **(c)** 218 **(d)** 294 **(e)** 327

   SOLUTION **(c)**: The cubes that are not on the surface form a smaller, $5 \times 5 \times 5$ cube, which means that there are 125 of them. Therefore there are $343 - 125 = 218$ cubed on the surface.

10. Each side of the cube depicted on the right is numbered with a positive integer in such a way that the products of the numbers on each pair of opposite sides are all the same. Find the lowest possible sum of all the numbers on the sides of the cube.



   (a) 78 **(b)** 80 **(c)** 89 **(d)** 107 **(e)** None of the above

   SOLUTION **(c)**: Since the products of the numbers on the opposite sides must be the same, they must be a multiple of 10, 14 and 15. The get the lowest possible sum, we must use the least common multiple. Since $10 = 2 \cdot 5$, $14 = 2 \cdot 7$ and $15 = 3 \cdot 5$, we need $2 \cdot 3 \cdot 5 \cdot 7 = 210$. So the number opposite to 10 is 21, opposite to 14 is 15, and opposite to 15 is 14. The sum of all the numbers is $10 + 21 + 14 + 15 + 15 + 14 = 89$.

11. Find the largest integer smaller than $\sqrt{22+\sqrt{22+\sqrt{22+\sqrt{22}}}}$.

    (a) 4 **(b)** 5 **(c)** 9 **(d)** 20 **(e)** 25

    SOLUTION **(b)**: First, $\sqrt{22+\sqrt{22+\sqrt{22+\sqrt{22}}}} > \sqrt{22+\sqrt{22}} > \sqrt{22+4} > 5$.
    Then, if we define a recursive sequence $a_0 = \sqrt{22}$, $a_{n+1} = \sqrt{22+a_n}$ for $n \geq 0$, our number is a_3. Clearly $a_0 < 6$. If $a_n < 6$, then $a_{n+1} = \sqrt{22+a_n} < \sqrt{22+6} < 6$. That means $a_n < 6$ for all $n \geq 0$. So our number is greater than 5, but less than 6.

12. Pat and Mat were trying to calculate the average of two numbers a and b using their calculator. First, Pat took the calculator and typed in $a + b \div 2$ and got 30. Then Mat took the same calculator and typed in $b + a \div 2$ and got 18. What was the correct average of the two numbers?

    (a) 28 **(b)** 24 **(c)** 16 **(d)** 12 **(e)** None of the above

    SOLUTION **(c)**: We know that

$$a + \frac{1}{2}b = 30$$
$$\frac{1}{2}a + b = 18$$

Adding both equations together gives us

$$\frac{3}{2}a + \frac{3}{2}b = 48.$$

Dividing both sides by 3 gives us

$$\frac{a+b}{2} = \frac{48}{3} = 16.$$

13. How many pairs of integers (x, y) are solutions of the equation

$$3x^2y - 10xy - 8y - 17 = 0?$$

(a) none **(b)** one **(c)** two **(d)** four **(e)** None of the above

SOLUTION **(b)**: We can rewrite the equation as

$$(3x^2 - 10x - 8)y = 17$$

If x is an integer, so is $3x^2 - 10x - 8$. There are only four ways to write 17 as a product of two integers: $17 \cdot 1$, $1 \cdot 17$, $(-17) \cdot (-1)$ and $(-1) \cdot (-17)$, which gives us four possibilities for $3x^2 - 10x - 8$:

- $3x^2 - 10x - 8 = 17$, or $3x^2 - 10x - 25 = 0$. The left side factors: $(3x + 5)(x - 5) = 0$, so we have an integer solution $x = 5$.
- $3x^2 - 10x - 8 = 1$, or $3x^2 - 10x - 9 = 0$. The left side does not factor using integers, there are no integer solutions.
- $3x^2 - 10x - 8 = -17$, or $3x^2 - 10x + 9 = 0$. The left side does not factor using integers, there are no integer solutions.
- $3x^2 - 10x - 8 = -1$, or $3x^2 - 10x - 7 = 0$. The left side does not factor using integers, there are no integer solutions.

Therefore the only pair of integer solutions is $(5, 1)$.

14. For how many integers is $\dfrac{11n + 14}{n - 2}$ an integer?

(a) 3 **(b)** 6 **(c)** 9 **(d)** 18 **(e)** None of the above

SOLUTION **(d)**: Rewrite

$$\frac{11n + 14}{n - 2} = \frac{11n - 22 + 22 + 14}{n - 2} = \frac{11(n - 2) + 36}{n - 2} = 11 + \frac{36}{n - 2}$$

The above will be an integer if and only if $n - 2$ is a factor of 36. Since $36 = 2^2 \cdot 3^2$, each factor of 36 looks like $\pm 1 \cdot 2^{k_2} \cdot 3^{k_3}$ where $k_2 = 0, 1, 2$ and $k_3 = 0, 1, 2$. That means there are 3×3 positive and 3×3 negative factors of 36. Altogether there are 18 of them.

15. A fair coin is tossed 100 times and it lands on heads all 100 times. What is the probability that it will land on heads on the 101st toss?

(a) 1 **(b)** $\frac{1}{2^{100}}$ **(c)** $\frac{1}{2^{101}}$ **(d)** $\frac{1}{2}$ **(e)** None of the above

SOLUTION **(d)**: For a fair coin, the probability that it lands on heads is always $\frac{1}{2}$, regardless of what the results of any previous tosses were. Of course if I saw a coin land on heads 100 times in a row, I would very much doubt it was a fair coin!

16. If two woodchucks would chuck 32 lb of wood in 12 min, how much wood would 3 woodchucks chuck in 8 min?

    (a) $21\frac{1}{3}$ lb **(b)** 32 lb **(c)** 64 lb **(d)** 80 lb **(e)** None of the above

    SOLUTION **(b)**: Assuming all woodchucks chuck wood at the same uniform speed:

- One woodchuck would chuck 16 lb of wood in 12 min.
- One woodchuck would chuck = of wood in 1 min.
- One woodchuck would chuck = of wood in 1 min.
- Three woodchucks would chuck 3 = 4 lb of wood in 1 min.
- Three woodchucks would chuck 8 = 32 lb of wood in 8 min.

17. Assuming that 'wigglers' are those who wiggle, 'wobblers' are those that wobble, and 'wagglers' are those who waggle, which of the following sets of premises will necessarily lead to the conclusion that "Wilbur is not a weeble"?

(a) All weebles wobble.
No wobblers wiggle.
Some wigglers waggle.
Wilbur waggles.

(b) Some weebles wobble.
No wobblers wiggle.
Some wagglers wiggle.
Wilbur waggles.

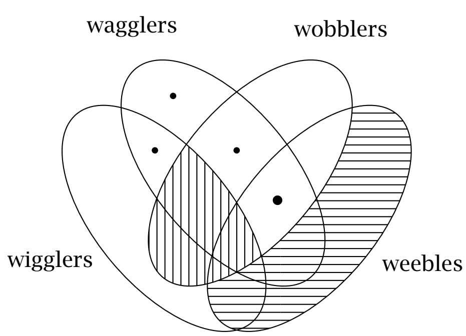
(c) All weebles wobble.
No wobblers wiggle.
All wigglers waggle.
Wilbur waggles.

(d) All weebles wobble.
No wobblers wiggle.
All wagglers wiggle.
Wilbur waggles.

(e) None of the above

Solution **(d)**: These are all arguments involving four sets: wigglers, wagglers, wobblers and weebles. We can visualize the premises using Venn diagrams with four sets. For each premise, we will shade the area that is guaranteed empty by that premise. Possible locations of Wilbur will be indicated by a dot.

First set of premises:



– "All weebles wobble." Shade the area of weebles that is ouside of wobblers with

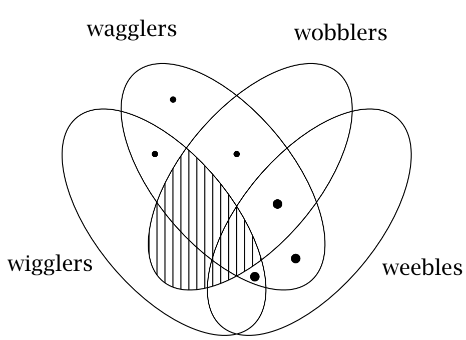
– "No wobblers wiggle." Shade the area of wobblers that is inside of wigglers with

– "Some wigglers waggle." This just guarantees that certain areas of the diagram will be non-empty, and has no relevance for Wilbur, as long as it is consistent with the other premises.

– "Wilbur waggles." There are four non-shaded areas inside the wagglers where Wilbur can be, as indicated by the dots.

We can see that Wilbur can be a weeble.

Second set of premises:



– "Some weebles wobble." This just guarantees that certain areas of the diagram will be non-empty, and has no relevance for Wilbur, as long as it is consistent with the other premises.

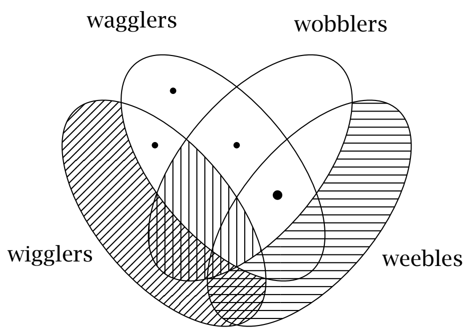
– "No wobblers wiggle." Shade the area of wobblers that is inside of wigglers with

– "Some wigglers waggle." This just guarantees that certain areas of the diagram will be non-empty, and has no relevance for Wilbur, as long as it is consistent with the other premises.

– "Wilbur waggles." There are six non-shaded areas inside the wagglers where Wilbur can be, as indicated by the dots.

We can see that Wilbur can be a weeble.

Third set of premises:



– "All weebles wobble." Shade the area of weebles that is ouside of wobblers with

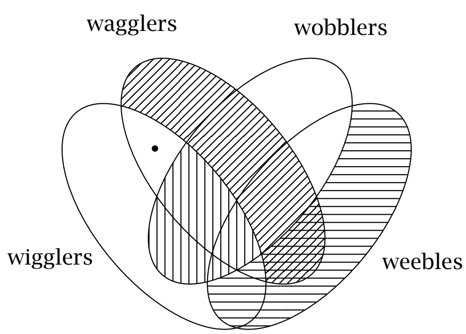
– "No wobblers wiggle." Shade the area of wobblers that is inside of wigglers with

– "All wigglers waggle." Shade the area of wigglers that is ouside of wagglers with

– "Wilbur waggles." There are four non-shaded areas inside the wagglers where Wilbur can be, as indicated by the dots. Some of these dots are inside weebles.

We can see that Wilbur can be a weeble.

Fourth set of premises:



– "All weebles wobble." Shade the area of weebles that is ouside of wobblers with

– "No wobblers wiggle." Shade the area of wobblers that is inside of wigglers with

– "All wagglers wiggle." Shade the area of wagglers that is ouside of wigglers with

– "Wilbur waggles." There is only one non-shaded area inside the wagglers where Wilbur can be, as indicated by the dot.

We can see that with this set of premises, Wilbur can not be a weeble.

18. Which of the expressions is equal to 0 for every x?

(a) $(x+1)(x-1)-x^2+1$ **(b)** x^0-1 **(c)** $\sqrt{x^2}-x$

(d) None of them is equal to 0 for every x. **(e)** More than one of them is 0 for every x.

SOLUTION **(a)**:

– $(x+1)(x-1)-x^2+1 = x^2-1-x^2+1 = 0$ for every x.

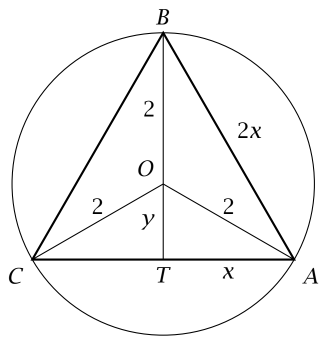
– x^0-1 is undefined when $x=0$.

– $\sqrt{x^2}-x$ is only equal to 0 when $x \geq 0$.

19. The vertices of an equilateral triangle lie on a circle with radius 2. The area of the triangle is

(a) $3\sqrt{3}$ **(b)** $2\sqrt{3}$ **(c)** $5\sqrt{3}$ **(d)** $4\sqrt{3}$ **(e)** None of the above

SOLUTION **(a)**: Draw the triangle and the circle. Lable the vertices of the triangle A, B and C, and the center of the circle O.



Let T be the midpoint of AC. Let $x = |TA|$ and $y = |TO|$.
The triangle $\triangle TAO$ is a right triangle, and so $x^2+y^2 = 4$. Similarly, the triangle $\triangle TAB$ is a right triangle, and so $x^2+(2+y)^2 = 4x^2$, or $(2+y)^2 = 3x^2$. Combining the equations together we get

$$\begin{aligned}(2+y)^2 &= 3(4-y^2)\\ (2+y)^2 &= 3(2+y)(2-y)\\ 2+y &= 3(2-y)\\ 2+y &= 6-3y\\ 4y &= 4\\ y &= 1\end{aligned}$$

Then $x = \sqrt{4-y^2} = \sqrt{3}$, and the area of the triangle is $A = x(y+2) = \sqrt{3}(1+2) = 3\sqrt{3}$.

20. How many 7-digit positive integers are made up of the digits 0 and 1 only and are divisible by 6?

(a) 10 **(b)** 11 **(c)** 16 **(d)** 21 **(e)** 33

SOLUTION **(b)**: Let n be a 7-digit positive integer made up of the digits 0 and 1 only, and that is divisible by 6. The leftmost digit of n cannot be 0, so must be 1.
Since n is divisible by 6, then n is even, which means that the rightmost digit of n cannot be 1, and so must be 0.
Therefore, n has the form $1pqrst0$ for some digits p, q, r, s, t each equal to 0 or 1.
Now, n is divisible by 6 exactly when it is divisible by 2 and by 3.
Since the ones digit of n is 0, then it is divisible by 2.
n is divisible by 3 exactly when the sum of its digits is divisible by 3. The sum of the digits of n is $1+p+q+r+s+t$. Since each of p, q, r, s, t is 0 or 1, then $1 \le 1+p+q+r+s+t \le 6$.
Thus, n is divisible by 3 exactly when $1+p+q+r+s+t$ is equal to 3 or to 6.
That is, n is divisible by 3 exactly when either 2 of p, q, r, s, t are 1s or all 5 of p, q, r, s, t are 1s.
There are 10 ways for 2 of these to be 1s.
These correspond to the pairs $pq, pr, ps, pt, qr, qs, qt, rs, rt, st$.
There is 1 way for all 5 of p, q, r, s, t to be 1s. Thus, there are $1+10 = 11$ such 7-digit integers.

21. Suppose that x and y satisfy $\frac{x-y}{x+y} = 9$ and $\frac{xy}{x+y} = -60$. The value of $(x+y)+(x-y)+xy$ is

(a) -50 **(b)** -150 **(c)** -14310 **(d)** 210 **(e)** 14160

SOLUTION **(b)**: Let $V = (x+y)+(x-y)+xy$. Then

$$\frac{V}{x+y} = \frac{x+y}{x+y} + \frac{x-y}{x+y} + \frac{xy}{x+y} = 1+9-60 = -50$$

and so $V = -50(x+y)$. Let $n = x+y$. Then $x-y = 9(x+y) = 9n$. Squaring both sides gives us $x^2 - 2xy + y^2 = 81n^2$. Since $x+y = n$, we also get $x^2+2xy+y^2 = n^2$. Subtracting the two equations will give us $-4xy = 80n^2$ or $xy = -20n^2$.
At the same time we know that $\frac{xy}{n} = -60$. So

$$-60 = \frac{xy}{n} = \frac{-20n^2}{n} = -20n$$

which means that $n = 3$.
Then $V = -50n = -150$.

22. A total of n points are equally spaced around a circle and are labeled with the integers 1 to n, in order. Two points are called diametrically opposite if the line segment joining them is a diameter of the circle. If the points labeled 7 and 35 are diametrically opposite, then n equals

(a) 54 **(b)** 55 **(c)** 56 **(d)** 57 **(e)** None of the above

SOLUTION **(c)**: The number of points on the circle equals the number of spaces between the points around the circle. Moving from the point labeled 7 to the point labeled 35 requires moving $35-7=28$ points and so 28 spaces around the circle.
Since the points labeled 7 and 35 are diametrically opposite, then moving along the circle from 7 to 35 results in traveling halfway around the circle.
Since 28 spaces makes half of the circle, then $2 \cdot 28 = 56$ spaces make the whole circle. Thus, there are 56 points on the circle, and so $n = 56$.

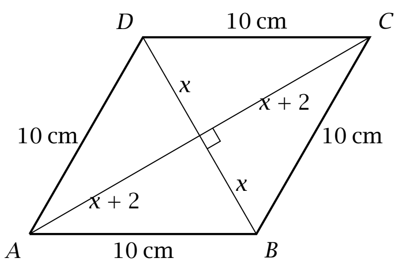
23. What is the area of a rhombus that has sides of length 10 cm and diagonals that differ in length by 4 cm?

(a) $96\,\text{cm}^2$ **(b)** $100\,\text{cm}^2$

(c) $100\sqrt{2}\,\text{cm}^2$ **(d)** Not enough information given.

(e) None of the above

SOLUTION **(a)**: Start by drawing the rhombus and the two diagonals.



Note that in a rhombus, the diagonals bisect each other and meet at a right angle. Label the length of each half of the shorter diagonal as x cm. Each half of the longer diagonal will then be $x + 2$ cm. The rhombus will then consists of 4 congruent right triangles, each with hypotenuse 10 cm and sides x cm and $x + 2$ cm. From the Pythagorean Theorem, $x^2 + (x + 2)^2 = 100$. This equation has one positive solution, $x = 6$. Then the area of each of the four right triangle that form the rhombus is $\frac{1}{2}x(x+2) = \frac{1}{2}\cdot 6\cdot 8 = 24\text{cm}^2$. The area of the rhombus is $4\cdot 24\text{cm}^2 = 96\text{cm}^2$.

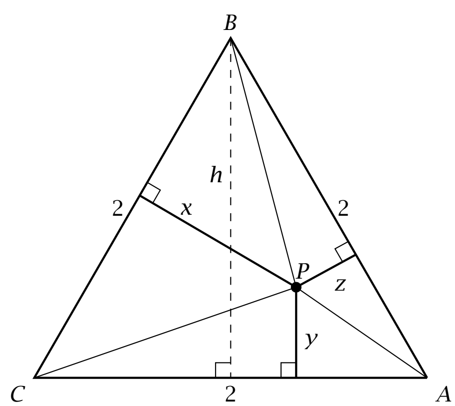
24. Point P lies inside an equilateral triangle whose sides are of length 2. If the distances from P to each side of the triangle are x, y and z, what is $x + y + z$?

(a) $\sqrt{3}$ **(b)** $2\sqrt{3}$

(c) $3\sqrt{3}$ **(d)** Not enough information given.

(e) None of the above

Solution **(a)**: The picture below shows an equilateral triangle $\triangle ABC$ with the point P.



The area of the $\triangle ABC$ is equal to

$$\frac{1}{2}2h = h = \sqrt{2^2 - 1^2} = \sqrt{3}$$

The same area can be computed as the sum of the areas of triangles $\triangle BPC$, $\triangle APC$ and $\triangle APB$.
The area of $\triangle APC$ is $\frac{1}{2}2y = y$.
The area of $\triangle BPC$ is $\frac{1}{2}2x = x$.
The area of $\triangle APB$ is $\frac{1}{2}2z = z$.
Therefore $x + y + z = h = \sqrt{3}$.

25. A rectangle is given with length l and width w, where $l > w$. One of them is increased by 20%, while the other is decreased by 20%. What happens to the area of the rectangle?

(a) It stays the same. **(b)** It always decreases.

(c) It always increases. **(d)** It increases only if the l is increased.

(e) It increases only if the w is increased.

Solution **(b)**: Let's call one of the dimensions a and the other b. Since the area is $A = ab$ and multiplication is commutative, it does not matter which one is which! If we increase a by 20%, the new dimension will be $1.2a$. If we decrease b by 20%, the new dimension will be $0.8b$. Then the new area will be $(1.2a)(0.8b) = 1.2 \cdot 0.8ab = 0.96ab$. The area will always decrease by 4%.