1. Give the exact value of

$$\sin^2\left(\frac{5\pi}{6}\right)+\sin^2\left(\frac{10\pi}{6}\right)+\sin^2\left(\frac{15\pi}{6}\right)+\sin^2\left(\frac{20\pi}{6}\right)+\cdots+\sin^2\left(\frac{500\pi}{6}\right)$$

(a) 49 **(b)** 50 **(c)** $50\frac{1}{4}$ **(d)** $50\frac{3}{4}$ **(e)** 51

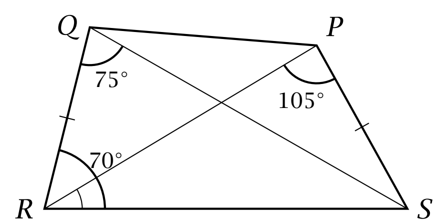
SOLUTION **(d)**: The first 6 terms are

$$\sin^2\left(\frac{5\pi}{6}\right)+\sin^2\left(\frac{10\pi}{6}\right)+\sin^2\left(\frac{15\pi}{6}\right)+\sin^2\left(\frac{20\pi}{6}\right)+\sin^2\left(\frac{25\pi}{6}\right)+\sin^2\left(\frac{30\pi}{6}\right)$$
$$=\frac{1}{4}+\frac{3}{4}+1+\frac{3}{4}+\frac{1}{4}+0=3$$

After that, the same 6 numbers repeat because of periodicity of sine. Since $\frac{500\pi}{6}=100\frac{5\pi}{6}$, there are 100 terms in the sum. Dividing by 6 gives us $100=16\cdot 6+4$, so the sum is

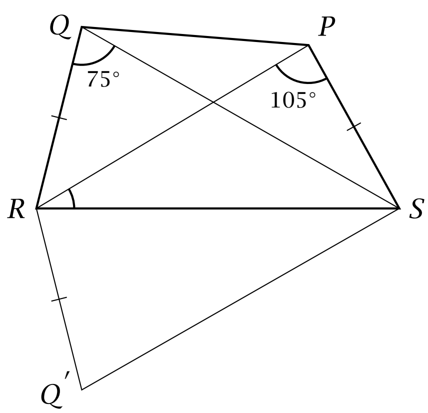
$$16\cdot 3+\frac{1}{4}+\frac{3}{4}+1+\frac{3}{4}=50\frac{3}{4}$$

2. In the quadrilateral $PQRS$, $\angle SPR=105°$, $\angle SQR=75°$, $\angle QRS=70°$, and $QR=PS$. Find $\angle PRS$.



(a) 15° **(b)** 25° **(c)** 35° **(d)** 45°

(e) None of the above



SOLUTION **(c)**: Reflect the point Q ovr the line $\overline{RS}$ to get a point Q'. Then $\angle RPS+\angle RQ'S=180$, so the quadrilateral $RPSQ'$ is cyclic. Moreover, since $RQ'=PS$, it is a cyclic trapezoid. Since every cyclic trapezoid is isosceles, $\angle PRQ'=\angle RPS=105°$. It is given that $\angle QRS=70°$, and therefore, from symmetry, $\angle Q'RS=70°$. Then $\angle PRS=\angle PRQ'-\angle SRQ'=105°-70°=35°$.

3. Which of the following functions has a graph with a horizontal asymptote?

I. $f(x) = \log_4(x + 2)$

II. $g(x) = \tan x$

III. $h(x) = 2e^x$

IV. $p(x) = \dfrac{x + x^2}{1 - x}$

V. $q(x) = \dfrac{x + x^2}{1 - x^2}$

(a) I, III, and IV only **(b)** I, III, and V only **(c)** II, III, and IV only

(d) III and IV only **(e)** III and V only

SOLUTION **(e)**: $f(x)$ has a vertical asymptote at $x = -2$. It is undefined for $x \le -2$, and it goes to ∞ as $x \to \infty$.
g is periodic with period π, so it cannot have a horizontal asymptote.
$h(x)$ has a horizontal asymptote $y = 0$ as $x \to -\infty$.
p is a rational function with degree of numerator larger than the degree of denominator, so it does not have a horizontal asymptote.
q is a rational function where the degree of numerator is the same as the degree of the denominator, so it has a horizontal asymptote.
So only the functions h and q have horizontal asymptotes.

4. Suppose you know that:

   ⊢ If thistles whistle, then weebles wobble.
   ⊢ Ants dance only if goggles giggle.
   ⊢ If thistles don't whistle, then ants dance.

   Which of the following must logically be true?

   (a) If weebles wobble, then goggles giggle.

   (b) Weebles don't wobble only if goggles giggle.

   (c) If ants dance, then weebles wobble.

   (d) Goggles giggle only if thistles whistle.

   (e) None of the above

   Solution **(b)**: Let w = weebles wobble, r = thistles whistle, p = ants dance, and q = goggles giggle. We are thus given:

$$r \to w$$
$$\sim q \to \sim p$$
$$\sim r \to p$$

The first premise $r \to w$ is equivalent to its contraposition: $\sim w \to \sim r$, so it can be replaced. Similarly, the second premise can be replaced by $p \to q$. Replacing and reordering, we get

$$\sim w \to \sim r$$
$$\sim r \to p$$
$$p \to q$$

which is equivalent by transitivity to $\sim w \to q$.
This can be translated to 'If weebles don't wobble, then goggles giggle.', which is equivalent to 'Weebles don't wobble only if goggles giggle.'

5. What is the sum of the solutions to the equation:

$$5^{2x^2+3x-7} = 13^{2x+5}$$

(a) $\dfrac{\ln\left(\frac{169}{125}\right)}{\ln 25}$ **(b)** $-\ln\left(\dfrac{5}{169}\right)$ **(c)** $-\dfrac{3}{2}+\ln\left(\dfrac{13}{5}\right)$

(d) $\dfrac{1}{2}\log 13$ **(e)** None of the above

Solution **(a)**: Since both sides of the equations are positive, we can apply the natural logarithm to both sides of the equation and get an equivalent equation

$$\ln\left(5^{2x^2+3x-7}\right) = \ln\left(13^{2x+5}\right)$$

Using properties of logarithm, this is equivalent to

$$(2x^2+3x-7)\ln 5 = (2x+5)\ln 13$$

This can be further manipulated:

$$\begin{aligned} 2x^2\ln 5 + 3x\ln 5 - 7\ln 5 &= 2x\ln 13 + 5\ln 13 \\ \ln\left(5^2\right)x^2 + \ln\left(5^3\right)x - \ln\left(5^7\right) &= x\ln\left(13^2\right)x + \ln\left(13^5\right) \\ \ln\left(5^2\right)x^2 + \ln\left(\frac{5^3}{13^2}\right)x - \ln\left(5^7\cdot 13^5\right) &= 0 \end{aligned}$$

This is a quadratic equation with positive leading coefficient and negative constant coefficient, so it has two real solutions. The sum of roots of a quadratic equation of the form $ax^2+bx+c=0$ is always $-{}^{b}\!/\!_{a}$, which in this case is

$$-\frac{\ln\left(\frac{5^3}{13^2}\right)}{\ln\left(5^2\right)} = \frac{\ln\left(\frac{13^2}{5^3}\right)}{\ln\left(5^2\right)} = \frac{\ln\left(\frac{169}{125}\right)}{\ln 25}$$

6. If we add all the solutions of the following equation, which number do we get?

$$\sqrt{1+\sqrt{x+\sqrt{6x-5}}}=\sqrt{2+\sqrt{x}}$$

(a) 10 **(b)** 9 **(c)** 5 **(d)** 1 **(e)** None of the above

SOLUTION **(b)**: First, since both sides of the equation are non-negative, we can safely square both sides:

$$\begin{aligned} 1+\sqrt{x+\sqrt{6x-5}} &= 2+\sqrt{x} \\ \sqrt{x+\sqrt{6x-5}} &= 1+\sqrt{x} \end{aligned}$$

Again, both sides of the last equation are non-negative, so we can square again:

$$\begin{aligned} x+\sqrt{6x-5} &= 1+2\sqrt{x}+x \\ \sqrt{6x-5} &= 1+2\sqrt{x} \end{aligned}$$

Again, we can safely square both sides:

$$\begin{aligned} 6x-5 &= 1+4\sqrt{x}+4x \\ 2x-6 &= 4\sqrt{x} \\ x-3 &= 2\sqrt{x} \end{aligned}$$

Now the left side of the last equation is no longer guaranteed to be non-negative, so if we square both sides again, we will have to check our solutions:

$$\begin{aligned} x^2-6x+9 &= 4x \\ x^2-10x+9 &= 0 \\ (x-9)(x-1) &= 0 \end{aligned}$$

The last equation has two solutions, $x = 1$ and $x = 9$. However, we have to check our solutions with the equation before the last squaring of both sides, since that could have introduced extraneous solutions.
Check $x = 1$: the left side is $1 - 3 = -2$, while the right side is $2\sqrt{1} = 2$. This is not a solution.
Check $x = 9$: the left side is $9 - 3 = 6$, while the right side is $2\sqrt{9} = 6$. This is a solution.
So 1 was an extraneous solution, and the only solution is 9.

7. Which of the following expressions are equivalent for all values of x?

I. $\dfrac{\tan^2 x}{\sec x - 1}$ II. $\dfrac{1 + \cos x}{\cos x}$

III. $\dfrac{1}{\sqrt{1 - \sin^2 x}} + 1$ IV. $\tan x \csc x + \sec^2 x - \tan^2 x$

(a) I and IV only **(b)** II and III and IV only

(c) I and II and IV only **(d)** All four are equivalent.

(e) No two of them are equivalent.

SOLUTION **(e)**: The expression

$$\frac{\tan^2 x}{\sec x - 1}$$

can be simplified as

$$\frac{\sec^2 x - 1}{\sec x - 1} = \frac{(\sec x + 1)(\sec x - 1)}{\sec x - 1} = \sec x + 1 \text{ if } \sec x \neq 1$$

The expression

$$\frac{1 + \cos x}{\cos x} = \frac{1}{\cos x} + 1 = \sec x + 1 \text{ for all } x$$

The expression

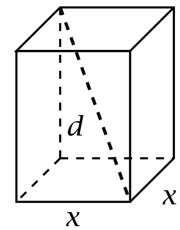
$$\frac{1}{\sqrt{1 - \sin^2 x}} + 1 = \frac{1}{\sqrt{\cos^2 x}} + 1 = \frac{1}{|\cos x|} + 1 = |\sec x| + 1$$

This is equal to $\sec x + 1$ only if $\sec x > 0$.
Finally,

$$\tan x \csc x + \sec^2 x - \tan^2 x = \frac{\sin x}{\cos x}\frac{1}{\sin x} + 1 = \sec x + 1 \text{ if } \sin x \neq 0$$

So while all of them are equal to $\sec x + 1$ for some values of x, they all have different exceptions, so no two of them are equivalent.

8. A rectangular box with square base of side length x ft, as shown, has volume $2\sqrt{2}$ ft^3. The diagonal d from the front, right, bottom corner to the back, left, top corner has length 1 foot longer than x. Which of the following must be true about x?



(a) $x^6 - 2x^5 - x^4 + 8 = 0$

(b) $2x^5 + x^4 - 8 = 0$ **(c)** $x^4 + 2x^3 + x^2 - 2\sqrt{2} = 0$

(d) $x^2 + (2 - \sqrt{2})x + 1 = 0$ **(e)** None of the above

SOLUTION **(a)**: Call the height of the box h. We know $hx^2 = 2\sqrt{2}$. The diagonal across the top of the box is $x\sqrt{2}$. Using the right triangle formed by h, d and the diagonal across the top of the box, we get $h^2 + 2x^2 = d^2$. We know that $d = x + 1$ and $h = \frac{2\sqrt{2}}{x^2}$. Plugging these in will give us

$$\frac{8}{x^4} + 2x^2 = x^2 + 2x + 1$$

This will simplify:

$$8 + 2x^6 = x^6 + 2x^5 + x^4$$
$$x^6 - 2x^5 - x^4 + 8 = 0$$

9. Which of the following numbers is the exact value of $\sin(\pi/12)$?

(a) $\dfrac{\sqrt{2}-1}{2}$ **(b)** $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ **(c)** $\dfrac{1}{2\sqrt{2}} - \dfrac{\sqrt{3}}{2\sqrt{2}}$

(d) $\dfrac{1}{\sqrt{6}+\sqrt{2}}$ **(e)** None of the above

SOLUTION **(d)**: Start by rewriting $\frac{\pi}{12}$ as $\frac{\pi}{4} - \frac{\pi}{6}$. Then

$$\begin{aligned}
\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\
&= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\
&= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2} \\
&= \frac{\sqrt{3}-1}{2\sqrt{2}} \\
&= \frac{(\sqrt{3}-1)(\sqrt{3}+1)}{2\sqrt{2}(\sqrt{3}+1)} \\
&= \frac{3-1}{2\sqrt{2}(\sqrt{3}+1)} \\
&= \frac{1}{\sqrt{2}(\sqrt{3}+1)} \\
&= \frac{1}{\sqrt{6}+\sqrt{2}}
\end{aligned}$$

10. Which of the following is a complete solution set to the equation: $\cos x + \sqrt{3}\sin x = 1$?

(a) $\{k\pi : k \text{ is an integer}\} \cup \{\frac{2\pi}{3} + k\pi : k \text{ is an integer}\}$

(b) $\{2k\pi : k \text{ is an integer}\} \cup \{\frac{2\pi}{3} + 2k\pi : k \text{ is an integer}\}$

(c) $\{2k\pi : k \text{ is an integer}\} \cup \{\pm\frac{2\pi}{3} + 2k\pi : k \text{ is an integer}\}$

(d) $\{k\pi : k \text{ is an integer}\} \cup \{\pm\frac{2\pi}{3} + 2k\pi : k \text{ is an integer}\}$

(e) None of the above

SOLUTION **(b)**: Trying $x = \pi$ shows that **(a)** and **(d)** are not correct.
Trying $x = -\frac{2\pi}{3}$ gives us

$$\cos\left(-\frac{2\pi}{3}\right) + \sqrt{3}\sin\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - \sqrt{3}\frac{\sqrt{3}2}{=} - 1 \neq 1$$

so **(c)** is also not correct. Trying $x = 0$ and $x = \frac{2\pi}{3}$ shows that the set in **(b)** is at least a subset of the solutions. To see that it must be a complete set:

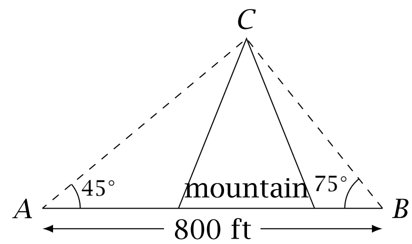
$$\begin{aligned}
\cos x + \sqrt{3}\sin x &= 1 && \text{subtract } \sqrt{3}\sin x \\
\cos x &= 1 - \sqrt{3}\sin x && \textbf{square both sides} \\
\cos^2 x &= 1 - 2\sqrt{3}\sin x + 3\sin^2 x && \text{Pythagorean identity} \\
1 - \sin^2 x &= 1 - 2\sqrt{3}\sin x + 3\sin^2 x && \text{subtract } 1 - \sin^2 x \\
0 &= 4\sin^2 x - 2\sqrt{3}\sin x
\end{aligned}$$

Solutions to the last equation are $x = k\pi$ for any integer k from $\sin x = 0$, or $x = \frac{\pi}{3} + 2k\pi$ or $x = \frac{2\pi}{3} + 2k\pi$ for any integer k from $\sin x = \frac{\sqrt{3}}{2}$. However, because we squared both sides of the equation, some of these may be extraneous solutions.
We already know that $2k\pi$ and $\frac{2\pi}{3} + 2k\pi$ work, and that $\pi + 2k\pi$ does not. All we need to check is $x = \frac{\pi}{3}$, but

$$\cos\left(\frac{\pi}{3}\right) + \sqrt{3}\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{3}{2} = 2 \neq 1$$

so $x = \frac{\pi}{3} + 2k\pi$ are also extraneous solutions, and the answer is **(b)**.

11. Point C is on the top of a mountain. The angle of elevation from a point A below the mountain to point C is 45°. From a point B, exactly 800 feet on the other side of the mountain from A, the angle of elevation to the point C is 75°. How high is the mountain?



(a) $\dfrac{800\sin(75^\circ)}{\sqrt{6}}$ **(b)** $\dfrac{200\sin(75^\circ)}{\sqrt{6}}$

(c) $800\sin(75^\circ)\sqrt{\dfrac{2}{3}}$ **(d)** $400\sin(75^\circ)\sqrt{\dfrac{2}{3}}$

(e) None of the above

Solution **(c)**: Subtracting from 180°, $\angle C = 60^\circ$. By the law of sines, If the side opposite to A is called a, then, by the law of sines,

$$\frac{a}{\sin 45^\circ} = \frac{800}{\sin 60^\circ}$$

and so

$$a = \frac{800\sqrt{2}}{\sqrt{3}}$$

If we call the height of the mountain h, then

$$h = a\sin 75^\circ = 800\sin(75^\circ)\sqrt{\frac{2}{3}}$$

12. If $f(x)$ is a function that satisfies $3f(x) = 2f(x-1) + f(x+1)$ for all x in the domain of f, which of the following functions could be $f(x)$?

(a) $f(x) = 12$ **(b)** $f(x) = 2^x$ **(c)** $f(x) = 8 - 2^{x-1}$

(d) $f(x) = \dfrac{3\,(2^{x+1}) - 5}{7}$ **(e)** All of the above

Solution **(e)**: For $f(x) = 12$, the right hand side is $2 \cdot 12 + 12 = 3 \cdot 12$.
For $f(x) = 2^x$, the right hand side is

$$2 \cdot 2^{x-1} + 2^{x+1} = 2 \cdot 2^x \cdot 2^{-1} + 2 \cdot 2^x = 2^x + 2 \cdot 2^x = 3 \cdot 2^x$$

At this moment we know the answer must be 'All of the above', but just for completeness, let's check the other two functions:
Both $8 - 2^{x-1}$ and $\frac{3(2^{x+1})-5}{7}$ can be written in the form $c_1 \cdot 2^x + c_2$. For any function in this form, the left hand side is

$$3c_1 \cdot 2^x + 3c_2$$

and the right hand side is

$$2c_1 \cdot 2^{x-1} + 2c_2 + c_1 \cdot 2^{x+1} + c + 2 = c_1 \cdot 2^x + 2c_1 \cdot 2^x + 3c_2$$

which simplifies to the right side.

13. Suppose you are on a game show where there are four doors, the prize is behind one of the doors and you are asked to select one at random. If you pick the door with the prize, you win. Once you have selected, the host opens one of the other three doors, which does not contain the prize. He then asks you if you want to stay with your first choice or select one of the remaining two doors.
Let P_1 be the probability of winning if you switch and pick one of the other two doors and P_2 be the probability of winning if you keep your first choice. What is $P_1 - P_2$?

(a) 0 **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{4}$ **(d)** $\frac{1}{8}$ **(e)** None of the above

Solution **(d)**: P_2 is $\frac{1}{4}$ because the probability that you win if you don't switch is the same as the probability that the first pick was right. For P_1 you need the probability that your first pick was wrong times the probability that your second pick was right. This is $(\frac{3}{4}) \times (\frac{1}{2}) = \frac{3}{8}$. So $P_1 - P_2 = \frac{3}{8} - \frac{1}{4} = \frac{1}{8}$.

14. Suppose a and b are positive real numbers and $\log_b(a+b) = 2.642$. What is the value of

$$\log_b\left(\frac{1}{\sqrt{a+b}-\sqrt{a}} - \frac{\sqrt{a}}{b}\right) \quad ?$$

(a) .321 **(b)** .6605 **(c)** 1.321 **(d)** 2.642

(e) There isn't enough information. The value of $\log_b a$ needs to be known.

SOLUTION **(a)**:

$$\begin{aligned}\frac{1}{\sqrt{a+b}-\sqrt{a}} - \frac{\sqrt{a}}{b} &= \frac{\left(\sqrt{a+b}+\sqrt{a}\right)}{\left(\sqrt{a+b}-\sqrt{a}\right)\left(\sqrt{a+b}+\sqrt{a}\right)} - \frac{\sqrt{a}}{b} \\ &= \frac{\sqrt{a+b}+\sqrt{a}}{a+b-a} - \frac{\sqrt{a}}{b} \\ &= \frac{\sqrt{a+b}}{b}\end{aligned}$$

So

$$\begin{aligned}\log_b\left(\frac{1}{\sqrt{a+b}-\sqrt{a}} - \frac{\sqrt{a}}{b}\right) &= \log_b\left(\frac{\sqrt{a+b}}{b}\right) \\ &= \frac{1}{2}\log_b(a+b) - \log_b b \\ &= \frac{2}{2}\cdot 2.642 - 1 = 1.321 - 1 = .321\end{aligned}$$

15. What is the exact value of $\cos^{-1}\left(\sin\left(\frac{13\pi}{12}\right)\right)$?

(a) $\frac{5\pi}{12}$ **(b)** $\frac{7\pi}{12}$ **(c)** $\frac{11\pi}{12}$ **(d)** $\frac{17\pi}{12}$ **(e)** None of the above

SOLUTION **(b)**: Let

$$\theta = \cos^{-1}\left(\sin\left(\frac{13\pi}{12}\right)\right)$$

Then

$$\cos\theta = \sin\left(\frac{13\pi}{12}\right)$$

and

$$0 \le \theta \le \pi$$

Using the cofunction identity,

$$\sin\left(\frac{13\pi}{12}\right) = \cos\left(\frac{\pi}{2} - \frac{13\pi}{12}\right) = \cos\left(\frac{-7\pi}{12}\right)$$

Unfortunately, $-\frac{7\pi}{12} \notin [0, \pi]$. However, since cosine is even,

$$\cos\left(-\frac{7\pi}{12}\right) = \cos\left(\frac{7\pi}{12}\right)$$

and $\frac{7\pi}{12} \in [0, \pi]$.

16. What is $\displaystyle\sum_{n=1}^{2019}\left(\frac{1+i}{\sqrt{2}}\right)^n$

(a) 0 **(b)** 1 **(c)** i **(d)** $\left(\sqrt{2}+1\right)i$

(e) None of the above

SOLUTION **(d)**: The exponential form of $\frac{1}{\sqrt{2}}(1+i)$ is $e^{i\pi/4}$, so the sum can be rewritten as

$$\sum_{n=1}^{2019} e^{i\frac{\pi}{4}n}$$

The function e^{it} is 2π-periodic, so

$$\sum_{n=1}^{8} e^{i\frac{\pi}{4}n} = \sum_{n=9}^{16} e^{i\frac{\pi}{4}n} = \sum_{n=17}^{24} e^{i\frac{\pi}{4}n} = \cdots$$

Furthermore,

$$e^{i\frac{\pi}{4}(n+4)} = e^{i\frac{\pi}{4}+i\pi} = e^{i\frac{\pi}{4}} \cdot e^{i\pi} = -e^{i\frac{\pi}{4}n}$$

so

$$\sum_{n=1}^{8} e^{i\frac{\pi}{4}n} = 0$$

Therefore

$$\sum_{n=1}^{2019} e^{i\frac{\pi}{4}n} = \sum_{n=2017}^{2019} e^{i\frac{\pi}{4}n} = \sum_{n=1}^{3} e^{i\frac{\pi}{4}n} = \frac{\sqrt{2}}{2}(1+i) + i + \frac{\sqrt{2}}{2}(-1+i) = \sqrt{2}i + i$$

17. A game has the following rules: An unbalanced coin is flipped until a head comes up. If the head comes up on the nth flip, you win 2^n dollars. The expected average payoff per game is 5 dollars. Find the probability of getting a head when flipping the coin once.

(a) $\frac{1}{2}$ **(b)** $\frac{5}{8}$ **(c)** $\frac{3}{8}$ **(d)** $\frac{3}{4}$ **(e)** None of the above

SOLUTION **(b)**: Let p be the probability of head when flipping the coin. Then the probability of tail is $1 - p$, and the probability P_n that the head comes up first time on the nth flip is the probability of $n - 1$ tails followed by 1 head, so

$$(1-p)^{n-1}p$$

Then the expected payoff per game is

$$\sum_{n=1}^{\infty} P_n 2^n = \sum_{n=1}^{\infty}(1-p)^{n-1}p2^n = \sum_{n=1}^{\infty} 2p\,(2\,(1-p))^{n-1}$$

he last is a geometric series with the first term $a_1 = 2p$ and the common ratio $r = 2(1-p)$. The sum of this series is

$$\frac{a_1}{1-r} = \frac{2p}{1-2(1-p)} = \frac{2p}{2p-1}$$

The expected payoff is 5 dollars, so

$$\begin{aligned} \frac{2p}{2p-1} &= 5 \\ 2p &= 10p - 5 \\ -8p &= -5 \\ p &= \frac{5}{8} \end{aligned}$$

18. How many positive integers n are there such that $\frac{n^3+6n+55}{n-2}$ is also an integer?

(a) 7 **(b)** 8 **(c)** 9 **(d)** 13 **(e)** There are infinitely many such values of n.

SOLUTION **(a)**: Using synthetic division

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 6 & 55 \\ & & 2 & 4 & 20 \\ \hline & 1 & 2 & 10 & 75 \end{array}$$

we see that

$$\frac{n^3+6n+55}{n-2} = n^2+2n+10+\frac{75}{n-2}$$

so

$$\frac{n^3+6n+55}{n-2}$$

is an integer if and only 75 is divisible by $n-2$.

The factors of 75 are ±1, ±3, ±5, ±15, ±25 and ±75, so $n-2$ must be one of these 12 numbers. However, since n must be positive, $n-2 \geq -1$, so only 7 of these numbers can be used. Therefore there are 7 positive numbers n such that

$$\frac{n^3+6n+55}{n-2}$$

is an integer.

19. You roll a fair six sided die 4 times. What is the probability that your 4 rolls form a strictly increasing sequence?

(a) $\frac{1}{4}$ **(b)** $\frac{5}{18}$ **(c)** $\frac{5}{72}$ **(d)** $\frac{5}{432}$ **(e)** None of the above

SOLUTION **(d)**: The number of ways to make four distinct rolls is $6 \cdot 5 \cdot 4 \cdot 3$, while the total number of ways to make four rolls is $6 \cdot 6 \cdot 6 \cdot 6$, so the probability of rolling 4 distinct numbers is

$$\frac{6 \cdot 5 \cdot 4 \cdot 3}{6 \cdot 6 \cdot 6 \cdot 6} = \frac{5}{18}.$$

For each 4 distinct rolls, there are 24 different ways to order them, and only one of the ways is strictly increasing, so the probability that a 4 given rolls are in an increasing order is $1/24$. Therefore the probability that the 4 rolls form an increasing sequence is

$$\frac{5}{18} \cdot \frac{1}{24} = \frac{5}{432}$$

20. What is $\log_5(6) \times \log_6(7) \times \log_7(8) \times \cdots \times \log_{3124}(3125)$?

(a) 625 **(b)** ln(625) **(c)** ln(3120) **(d)** 5 **(e)** None of the above

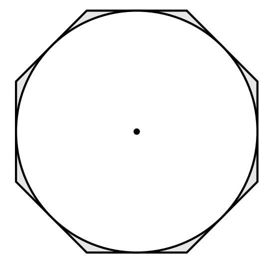
SOLUTION **(d)**: By the base change formula,

$$\log_b a = \frac{\ln a}{\ln b}$$

so the product is a telescoping product

$$\frac{\ln 6}{\ln 5} \cdot \frac{\ln 7}{\ln 6} \cdot \frac{\ln 8}{\ln 7} \cdot \cdots \cdot \frac{\ln 3124}{\ln 3123} \cdot \frac{\ln 3125}{\ln 3124} = \frac{\ln 3125}{\ln 5} = \log_5 3125 = 5$$

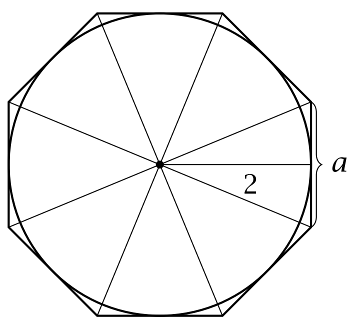
21. A circle of radius 2 is inscribed in a regular octagon. Find the area of the shaded region.



(a) $4(2 + \sqrt{8} - \pi)$

(b) $2\sqrt{2} - 4\pi$ **(c)** $\frac{28}{3}\sqrt{2} - 4\pi$

(d) $\frac{32}{1+\sqrt{2}} - 4\pi$ **(e)** None of the above



SOLUTION **(d)**: The octagon consists of 8 isosceles triangles with central vertex angle 45° and altitude 2, as shown. Let a be the length of one side of the octagon. Then

$$\frac{a}{2} = 2\tan\frac{45^\circ}{2}$$

Using a half angle formula for tangent:

$$\frac{a}{2} = 2\frac{\sin 45^\circ}{1 + \cos 45^\circ} = 2\frac{\frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} = \frac{2}{1 + \sqrt{2}}$$

The area of each of the 8 isosceles triangles is then

$$\frac{1}{2}ah = \frac{2}{1 + \sqrt{2}} \cdot 2 = \frac{4}{1 + \sqrt{2}}$$

The area of the octagon is then

$$\frac{32}{1 + \sqrt{2}}$$

and the shaded are is the area of the octagon minus the area of the circle:

$$\frac{32}{1 + \sqrt{2}} - 4\pi$$

22. Let a be a non-negative real number such that $f(x) = \dfrac{1}{a} - \dfrac{1}{a^x+1}$ is an odd function. Then the range of f is:

(a) $(-\infty, 0)$

(b) $(-1, 1)$

(c) $(-2, 2)$

(d) $(-½, ½)$

(e) $f(x)$ cannot be an odd function.

Solution **(d)**: First, $(-\infty, 0)$ cannot be a range of an odd function, so we can rule that one out. Second, regardless of a,

$$\frac{1}{a^x+1}$$

is always between 0 and 1, and for if $a > 0$,

$$\frac{1}{a^x+1} \to 1 \text{ as } x \to -\infty$$
$$\frac{1}{a^x+1} \to 0 \text{ as } x \to \infty$$

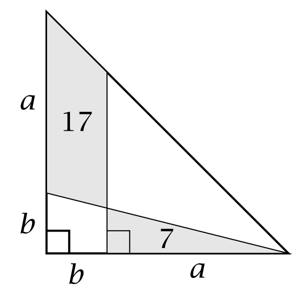
so the only way for f to be odd is when $\frac{1}{a} = \frac{1}{2}$, in which case the range is $(-½, ½)$. The only thing we need to verify is that

$$f(x) = \frac{1}{2} - \frac{1}{2^x+1}$$

is an odd function.

$$f(-x) = \frac{1}{2} - \frac{1}{2^{-x}+1} = \frac{2^{-x}-1}{2\,(2^{-x}+1)}\frac{2^x}{2^x} = \frac{1-2^x}{2\,(2^x+1)} = -\frac{2^x-1}{2\,(2^x+1)} = -\frac{2^x+1-2}{2\,(2^x+1)} = -f(x)$$

23. The two shaded areas in the right triangle are 17 and 7, as marked. Find ab.



(a) 20 **(b)** 24 **(c)** 48 **(d)** $\sqrt{119}$

(e) None of the above

SOLUTION **(a)**: The small trapezoidal area in the lower left corner of the triangle can be written in two different ways:

$$\frac{1}{2}(a+b+a)b-17$$

or

$$\frac{1}{2}(a+b)b-7$$

Setting these equal to each other and solving:

$$\begin{aligned}
\frac{1}{2}(a+b+a)b-17 &= \frac{1}{2}(a+b)b-7 \\
\frac{1}{2}(2a+b)b &= \frac{1}{2}(a+b)b+10 \\
2ab+b^2 &= ab+b^2+20 \\
ab &= 20
\end{aligned}$$

24. Suppose that

$$1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots = 2018 \quad \text{and} \quad 1 + \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3} + \cdots = 2019$$

What is pq?

(a) $\frac{2018}{2017}$ **(b)** $\frac{2019}{2017}$ **(c)** $\frac{2019}{2018}$ **(d)** $\frac{2017}{2019}$ **(e)** $\frac{2018}{2019}$

SOLUTION **(b)**: First,

$$\begin{aligned} 2018 &= \frac{1}{1 - \frac{1}{p}} \\ 1 - \frac{1}{p} &= \frac{1}{2018} \\ \frac{1}{p} &= 1 - \frac{1}{2018} = \frac{2017}{2018} \\ p &= \frac{2018}{2017} \end{aligned}$$

Similarly,

$$q = \frac{2019}{2018}$$

Then

$$pq = \frac{2018}{2017} \cdot \frac{2019}{2018} = \frac{2019}{2017}$$

25. A game is played by rolling a fair 6-sided die. If a 6 is rolled, the player wins. If a 2, 3 or 5 is rolled, the game stops and the player loses. If a 1 or 4 is rolled, the player gets to roll again. What is the probability that the player will eventually win?

(a) $1/6$ **(b)** $1/4$ **(c)** $1/3$ **(d)** $2/3$ **(e)** None of the above

SOLUTION **(b)**: We add up all the ways the player can win. The probability that the player wins on the first roll is $1/6$. The probability that the player can win on the second roll is the probability that the first roll is 1 or 4 and the second roll is a 6, so it's $(1/3) \times (1/6)$. Continuing with the same reasoning, the probability that the player wins on the nth roll is $(1/3)^n \times (1/6)$. So the sum of all the ways the player can win is an infinite geometric series with first term $1/6$ and r = $1/3$, so it converges to

$$\frac{1/6}{1 - 1/3} = 1/4.$$