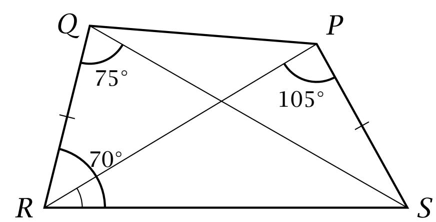
Saginaw Valley State University
2019 Math Olympics — Level II

1. Give the exact value of

$$\sin^2\left(\frac{5\pi}{6}\right) + \sin^2\left(\frac{10\pi}{6}\right) + \sin^2\left(\frac{15\pi}{6}\right) + \sin^2\left(\frac{20\pi}{6}\right) + \cdots + \sin^2\left(\frac{500\pi}{6}\right)$$

(a) 49 **(b)** 50 **(c)** $50\frac{1}{4}$ **(d)** $50\frac{3}{4}$ **(e)** 51

2. In the quadrilateral $PQRS$, $\angle SPR = 105^\circ$, $\angle SQR = 75^\circ$, $\angle QRS = 70^\circ$, and $QR = PS$. Find $\angle PRS$.



(a) 15° **(b)** 25° **(c)** 35° **(d)** 45°

(e) None of the above

3. Which of the following functions has a graph with a horizontal asymptote?

I. $f(x) = \log_4(x+2)$

II. $g(x) = \tan x$

III. $h(x) = 2e^x$

IV. $p(x) = \dfrac{x+x^2}{1-x}$

V. $q(x) = \dfrac{x+x^2}{1-x^2}$

(a) I, III, and IV only **(b)** I, III, and V only **(c)** II, III, and IV only

(d) III and IV only **(e)** III and V only

4. Suppose you know that:

⊢ If thistles whistle, then weebles wobble.
⊢ Ants dance only if goggles giggle.
⊢ If thistles don't whistle, then ants dance.

Which of the following must logically be true?

(a) If weebles wobble, then goggles giggle.

(b) Weebles don't wobble only if goggles giggle.

(c) If ants dance, then weebles wobble. **(d)** Goggles giggle only if thistles whistle.

(e) None of the above

5. What is the sum of the solutions to the equation:

$$5^{2x^2+3x-7} = 13^{2x+5}$$

(a) $\dfrac{\ln\left(\frac{169}{125}\right)}{\ln 25}$ **(b)** $-\ln\left(\dfrac{5}{169}\right)$ **(c)** $-\dfrac{3}{2}+\ln\left(\dfrac{13}{5}\right)$

(d) $\dfrac{1}{2}\log 13$ **(e)** None of the above

6. If we add all the solutions of the following equation, which number do we get?

$$\sqrt{1+\sqrt{x+\sqrt{6x-5}}} = \sqrt{2+\sqrt{x}}$$

(a) 10 **(b)** 9 **(c)** 5 **(d)** 1 **(e)** None of the above

7. Which of the following expressions are equivalent for all values of x?

I. $\dfrac{\tan^2 x}{\sec x - 1}$ II. $\dfrac{1+\cos x}{\cos x}$

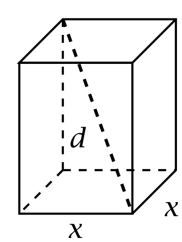
III. $\dfrac{1}{\sqrt{1-\sin^2 x}} + 1$ IV. $\tan x \csc x + \sec^2 x - \tan^2 x$

(a) I and IV only **(b)** II and III and IV only

(c) I and II and IV only **(d)** All four are equivalent.

(e) No two of them are equivalent.

8. A rectangular box with square base of side length x ft, as shown, has volume $2\sqrt{2}$ ft^3. The diagonal d from the front, right, bottom corner to the back, left, top corner has length 1 foot longer than x. Which of the following must be true about x?



(a) $x^6 - 2x^5 - x^4 + 8 = 0$

(b) $2x^5 + x^4 - 8 = 0$ **(c)** $x^4 + 2x^3 + x^2 - 2\sqrt{2} = 0$

(d) $x^2 + (2-\sqrt{2})x + 1 = 0$ **(e)** None of the above

9. Which of the following numbers is the exact value of $\sin(\pi/12)$?

(a) $\dfrac{\sqrt{2}-1}{2}$ **(b)** $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ **(c)** $\dfrac{1}{2\sqrt{2}} - \dfrac{\sqrt{3}}{2\sqrt{2}}$

(d) $\dfrac{1}{\sqrt{6}+\sqrt{2}}$ **(e)** None of the above

10. Which of the following is a complete solution set to the equation: $\cos x + \sqrt{3}\sin x = 1$?

(a) $\{k\pi : k \text{ is an integer}\} \cup \{\frac{2\pi}{3} + k\pi : k \text{ is an integer}\}$

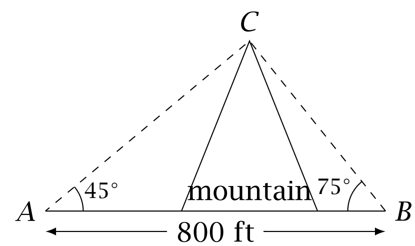
(b) $\{2k\pi : k \text{ is an integer}\} \cup \{\frac{2\pi}{3} + 2k\pi : k \text{ is an integer}\}$

(c) $\{2k\pi : k \text{ is an integer}\} \cup \{\pm\frac{2\pi}{3} + 2k\pi : k \text{ is an integer}\}$

(d) $\{k\pi : k \text{ is an integer}\} \cup \{\pm\frac{2\pi}{3} + 2k\pi : k \text{ is an integer}\}$

(e) None of the above

11. Point C is on the top of a mountain. The angle of elevation from a point A below the mountain to point C is 45°. From a point B, exactly 800 feet on the other side of the mountain from A, the angle of elevation to the point C is 75°. How high is the mountain?



(a) $\dfrac{800\sin(75°)}{\sqrt{6}}$ **(b)** $\dfrac{200\sin(75°)}{\sqrt{6}}$

(c) $800\sin(75°)\sqrt{\dfrac{2}{3}}$ **(d)** $400\sin(75°)\sqrt{\dfrac{2}{3}}$

(e) None of the above

12. If $f(x)$ is a function that satisfies $3f(x) = 2f(x-1) + f(x+1)$ for all x in the domain of f, which of the following functions could be $f(x)$?

(a) $f(x) = 12$ **(b)** $f(x) = 2^x$ **(c)** $f(x) = 8 - 2^{x-1}$

(d) $f(x) = \dfrac{3\left(2^{x+1}\right) - 5}{7}$ **(e)** All of the above

13. Suppose you are on a game show where there are four doors, the prize is behind one of the doors and you are asked to select one at random. If you pick the door with the prize, you win. Once you have selected, the host opens one of the other three doors, which does not contain the prize. He then asks you if you want to stay with your first choice or select one of the remaining two doors.
Let P_1 be the probability of winning if you switch and pick one of the other two doors and P_2 be the probability of winning if you keep your first choice. What is $P_1 - P_2$?

(a) 0 **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{4}$ **(d)** $\frac{1}{8}$ **(e)** None of the above

14. Suppose a and b are positive real numbers and $\log_b(a+b) = 2.642$. What is the value of

$$\log_b\left(\frac{1}{\sqrt{a+b}-\sqrt{a}} - \frac{\sqrt{a}}{b}\right) \quad ?$$

(a) .321 **(b)** .6605 **(c)** 1.321 **(d)** 2.642

(e) There isn't enough information. The value of $\log_b a$ needs to be known.

15. What is the exact value of $\cos^{-1}\left(\sin\left(\frac{13\pi}{12}\right)\right)$?

(a) $\frac{5\pi}{12}$ **(b)** $\frac{7\pi}{12}$ **(c)** $\frac{11\pi}{12}$ **(d)** $\frac{17\pi}{12}$ **(e)** None of the above

16. What is $\sum_{n=1}^{2019}\left(\frac{1+i}{\sqrt{2}}\right)^n$

(a) 0 **(b)** 1 **(c)** i **(d)** $\left(\sqrt{2}+1\right)i$

(e) None of the above

17. A game has the following rules: An unbalanced coin is flipped until a head comes up. If the head comes up on the nth flip, you win 2^n dollars. The expected average payoff per game is 5 dollars. Find the probability of getting a head when flipping the coin once.

(a) $\frac{1}{2}$ **(b)** $\frac{5}{8}$ **(c)** $\frac{3}{8}$ **(d)** $\frac{3}{4}$ **(e)** None of the above

18. How many positive integers n are there such that $\frac{n^3+6n+55}{n-2}$ is also an integer?

(a) 7 **(b)** 8 **(c)** 9 **(d)** 13 **(e)** There are infinitely many such values of n.

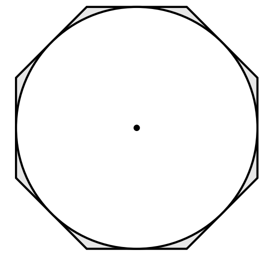
19. You roll a fair six sided die 4 times. What is the probability that your 4 rolls form a strictly increasing sequence?

(a) $\frac{1}{4}$ **(b)** $\frac{5}{18}$ **(c)** $\frac{5}{72}$ **(d)** $\frac{5}{432}$ **(e)** None of the above

20. What is $\log_5(6) \times \log_6(7) \times \log_7(8) \times \cdots \times \log_{3124}(3125)$?

(a) 625 **(b)** ln(625) **(c)** ln(3120) **(d)** 5 **(e)** None of the above

21. A circle of radius 2 is inscribed in a regular octagon. Find the area of the shaded region.



(a) $4(2 + \sqrt{8} - \pi)$

(b) $2\sqrt{2} - 4\pi$ **(c)** $\frac{28}{3}\sqrt{2} - 4\pi$

(d) $\frac{32}{1+\sqrt{2}} - 4\pi$ **(e)** None of the above

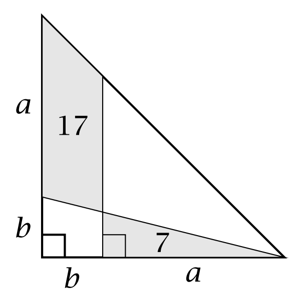
22. Let a be a non-negative real number such that $f(x) = \dfrac{1}{a} - \dfrac{1}{a^x + 1}$ is an odd function. Then the range of f is:

(a) $(-\infty, 0)$ **(b)** $(-1, 1)$

(c) $(-2, 2)$ **(d)** $(-\frac{1}{2}, \frac{1}{2})$

(e) $f(x)$ cannot be an odd function.

23. The two shaded areas in the right triangle are 17 and 7, as marked. Find ab.



(a) 20 **(b)** 24 **(c)** 48 **(d)** $\sqrt{119}$

(e) None of the above

24. Suppose that

$$1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots = 2018 \text{ and } 1 + \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3} + \cdots = 2019$$

What is pq?

(a) $\dfrac{2018}{2017}$ **(b)** $\dfrac{2019}{2017}$ **(c)** $\dfrac{2019}{2018}$ **(d)** $\dfrac{2017}{2019}$ **(e)** $\dfrac{2018}{2019}$

25. A game is played by rolling a fair 6-sided die. If a 6 is rolled, the player wins. If a 2, 3 or 5 is rolled, the game stops and the player loses. If a 1 or 4 is rolled, the player gets to roll again. What is the probability that the player will eventually win?

(a) $\frac{1}{6}$ **(b)** $\frac{1}{4}$ **(c)** $\frac{1}{3}$ **(d)** $\frac{2}{3}$ **(e)** None of the above