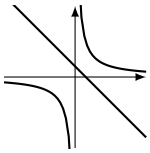
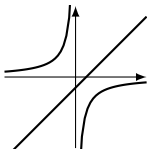
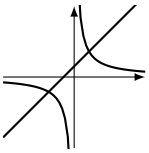
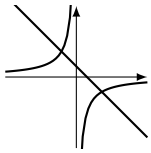
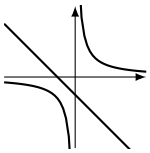
Saginaw Valley State University

# 2019 Math Olympics — Level I

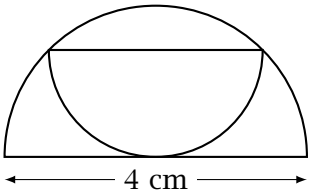
1. Jacob wanted to get rid of most of his toy cars. He gave one-sixth of his collection, plus 8 cars, to Xu, and one-fourth of what was left, plus 4 cars, to Olivia. Then he gave half of what was left to Ahmed. Then he gave 40% of what was left, plus 5 cars, to Jim. He kept the last 10 cars. How many toy cars did Jacob have at the beginning?

   **(a)** 116 **(b)** 106 **(c)** 96 **(d)** 86 **(e)** None of the above

2. Let $k$ be a non-zero real number. Two functions, $f(x) = \frac{k}{x}$ and $g(x) = kx + k$, were plotted on the same coordinate plane. Which of the following is the correct plot?

   **(a)**  **(b)**  **(c)**  **(d)**  **(e)** 

3. A small semi-circle is inscribed in a large semi-circle as shown on the right. What is the radius of the small semi-circle?

   **(a)** $\frac{\sqrt{2}}{2}$ cm **(b)** $\sqrt{2}$ cm **(c)** 1.5 cm **(d)** 2 cm

   **(e)** None of the above

4. Suppose you know that:

   - ⊢ If thistles whistle, then weebles wobble.
   - ⊢ Ants dance only if goggles giggle.
   - ⊢ If thistles don't whistle, then ants dance.

   Which of the following must logically be true?

   **(a)** If weebles wobble, then goggles giggle.

   **(b)** Weebles don't wobble only if goggles giggle.

   **(c)** If ants dance, then weebles wobble. **(d)** Goggles giggle only if thistles whistle.

   **(e)** None of the above

5. To simplify your life, you decide to only buy red socks and green socks. This worked so well that you were able to devote your time to other matters, and soon you can afford to hire a housekeeper from the cheapest agency in town. You and your brother come home one day, and the tired housekeeper tells you that he tried to organize your socks for you, but he couldn't fit all the red socks in one drawer or all the green socks in one drawer. Thus, he filled one drawer with red socks, one with green socks, and placed the rest of both red and green socks in another drawer. He labeled the top drawer of your three-drawer dresser 'red only', the middle one 'green only', and the bottom one 'both'. The housekeeper then asks if anyone has seen his glasses before trudging home. Your brother checks the drawers and bursts out laughing. He tells you that although the housekeeper did sort the socks as described, not a single drawer is labeled correctly. He then bets you \$100 that you can't correctly relabel the drawers after choosing just one of the drawers and having him hand you just one sock out of that drawer. You need the money so that you can hire a new housekeeper! Which of the following is true?

   **(a)** The only way to correctly relabel the drawers is to choose the top drawer (labeled 'red only').

   **(b)** The only way to correctly relabel the drawers is to choose the middle drawer (labeled 'green only').

   **(c)** The only way to correctly relabel the drawers is to choose the bottom drawer (labeled 'both').

   **(d)** You can correctly relabel the drawers no matter which drawer you pick.

   **(e)** You should not take the bet because, no matter which drawer you pick, you will not have enough information to correctly relabel the drawers.

6. Which of the following numbers is different from the rest?

   **(a)** $\dfrac{1+\sqrt{5}}{2}$ **(b)** $\dfrac{(1+\sqrt{5})^2}{4}-1$ **(c)** $\sqrt{\dfrac{3+\sqrt{5}}{2}}$

   **(d)** $\dfrac{2}{\sqrt{5}-1}$ **(e)** They are all the same.

7. Find the area of the triangle bounded by the $x$-axis and the lines $2y = 3x$ and $y = -3x + 9$.

   **(a)** 4.5 **(b)** 3 **(c)** 13.5 **(d)** 9 **(e)** None of the above

8. Let $z$ and $w$ be complex numbers, and for any complex number $x$, let $\overline{x}$ denote the complex conjugate of $x$. Which of the following must be a real number?

   **(a)** $z(\overline{z}+w)+\overline{zw}$ **(b)** $z\overline{w}-\overline{z}w$ **(c)** $z+\overline{\overline{z}}$ **(d)** $(z-\overline{z})(z+\overline{z})$

   **(e)** Either more than one or none of them are real numbers.

9. Which of the following is the domain of the function

$$f(x) = \frac{\sqrt{x^2 - 3x - 4}}{\sqrt[3]{x(x+5)}} \ ?$$

**(a)** $(-\infty, -5) \cup [-1, 0) \cup [4, \infty)$ **(b)** $(-\infty, -5) \cup (-1, 0) \cup (4, \infty)$

**(c)** $(-\infty, -5) \cup (-5, -1] \cup [4, \infty)$ **(d)** $(-\infty, -5) \cup [4, \infty)$

**(e)** None of the above

10. Solve the inequality $\dfrac{1}{x-3} \leq 1$

**(a)** $(-\infty, 3) \cup [4, \infty)$ **(b)** $(-\infty, 4]$ **(c)** $[4, \infty)$

**(d)** $(3, 4]$ **(e)** None of the above

11. Let $f$ be a function and let $g(x) = 2\,|f(3-x)| - 1$. Suppose that $(1, -5)$ is a point on the graph of $f$. Which of the following points must be on the graph of $g$?

**(a)** $(2, 9)$ **(b)** $(2, 12)$ **(c)** $(-4, 9)$ **(d)** $(-4, 12)$ **(e)** None of the above

12. If 3½ children can make 2½ snowmen in 1½ days, how many days will it take $x$ children to make $y$ snowmen?

**(a)** $\dfrac{5y}{7x}$ **(b)** $\dfrac{10xy}{21}$ **(c)** $\dfrac{10y}{21x}$ **(d)** $\dfrac{21y}{10x}$ **(e)** None of the above

13. A super amphitheater has 91 seats in the bottom row, 93 seats in the next row, 95 seats in the next row, and so on up to the top. If the total number of seats in the theater is 22,000, how many rows are there?

**(a)** 220 **(b)** 200 **(c)** 110 **(d)** 260 **(e)** None of the above

14. Mike can bike three times as fast as he can run. He enters a two-part biking-running race that is a total distance of 120 miles over a total time of 4 hours. If there is no time break between the two parts, and he can run one mile in 5 minutes, how long did he bike?

**(a)** 1 hour **(b)** 2 hours **(c)** 3 hours **(d)** 4.5 hours **(e)** None of the above

15. In how many different ways can 8 mathematicians and 5 physicists stand in a line if no two physicists can stand next to each other?

    **(a)** $13! - (8!/3!)$ **(b)** $(8! \times 9!)/4!$ **(c)** $8! \times 5! \times {}_8C_5$

    **(d)** $8! \times {}_9C_5$ **(e)** None of the above

16. A tennis tournament is played between 2 teams, team $A$ and team $B$. A winner is declared if a team wins three games in a row or four games. If we assume no ties, and we know that team $A$ has won the first two games, how many different ways are there for the tournament to finish? (For example, $A$ wins, $A$ wins, $B$ wins $B$ wins $A$ wins $A$ wins is one way, since team $A$ has won 4 times.)

    **(a)** 9 **(b)** 16 **(c)** 32 **(d)** 8 **(e)** None of the above

17. Which of the following functions is not even?

    **(a)** $f(x) = (x^2 - 1)^3$ **(b)** $g(x) = (x^3 - x)^2$ **(c)** $h(x) = (x^3 - 1)^2$

    **(d)** $k(x) = (x^2 - 1)^2$ **(e)** They are all even.

18. Let $x \oplus y$ be defined by $x \oplus y = \dfrac{x}{y} + \dfrac{y}{x}$ where $x$ and $y$ are non zero real numbers. Which of the following are true?

    **(a)** $\oplus$ is associative but not commutative **(b)** $\oplus$ is commutative but not associative

    **(c)** $\oplus$ is both associative and commutative **(d)** $\oplus$ is neither associative nor commutative

    **(e)** 1 is the identity for $\oplus$

19. How many positive integers $n$ are there such that $\dfrac{n^3 + 6n + 55}{n - 2}$ is also an integer?

    **(a)** 7 **(b)** 8 **(c)** 9 **(d)** 13 **(e)** There are infinitely many such values of $n$.

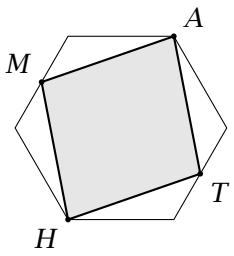
20. A mad scientist does all her calculations in a base 26 number system, where the letters of the English alphabet, in the alphabetical order, are used as digits, that is, $A$ represents 0, $B$ represents 1, …, $Y$ represents 24, and $Z$ represents 25.
    The full table is below:

| $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ | $I$ | $J$ | $K$ | $L$ | $M$ | $N$ | $O$ | $P$ | $Q$ | $R$ | $S$ | $T$ | $U$ | $V$ | $W$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |

    In this system, what is $ONE + TWO$?

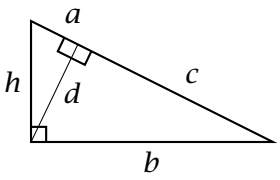
    **(a)** $IKS$ **(b)** $BIKS$ **(c)** $BIJS$ **(d)** $BIJE$ **(e)** None of the above

21. In the diagram, the points $M$ and $T$ are the midpoints of the opposite sides of a regular hexagon. What fraction of the hexagon is shaded?




**(a)** $\frac{2}{3}$ **(b)** $\frac{3}{4}$ **(c)** $\frac{5}{4}$ **(d)** $\frac{7}{8}$ **(e)** $\frac{8}{9}$

22. Given that $h = 4a$, which of the following does $ac$ equal?



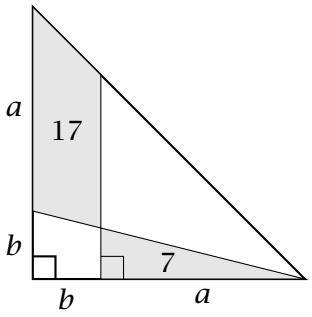

**(a)** $\frac{1}{2}b^2$ **(b)** $\frac{1}{4}b^2$ **(c)** $\frac{1}{8}b^2$ **(d)** $\frac{1}{16}b^2$

**(e)** None of the above

23. Suppose a 5-bit codeword is transmitted 5 times, once with 4 errors, once with three errors, once with 2 errors, once with 1 error, and once correctly (not necessarily in that order). The 5 received codewords are given below. Which one is the correct codeword?

**(a)** 00101 **(b)** 11010 **(c)** 11100 **(d)** 10100 **(e)** 00001

24. Two shaded areas in the right triangle are 17 and 7, as marked. Find $ab$.




**(a)** 20 **(b)** 24 **(c)** 48 **(d)** $\sqrt{119}$

**(e)** None of the above

25. Let $f : \mathbb{R} \to \mathbb{R}$ be an odd periodic function with period 5. Given that $f(7) = 9$, what is $f(2020) - f(2018)$?

**(a)** 6 **(b)** 7 **(c)** 8 **(d)** 9 **(e)** None of the above