Saginaw Valley State University

2018 Math Olympics — Level II Solutions

1. Which of the following is the value of $\frac{2}{3}+\frac{1}{4}+\frac{2}{15}+\frac{1}{12}+\frac{2}{35}+\frac{1}{24}+\frac{2}{63}\cdots+\frac{2}{(2017)(2019)}$?

   (a) $1-\frac{1}{(2018)(2019)}$ **(b)** $\frac{3}{2}-\frac{4037}{(2018)(2019)}$ **(c)** $1+\frac{1}{(2018)(2019)}$

   (d) $\frac{2018}{2019}$ **(e)** None of the above

   SOLUTION **(b)**:

$$\frac{2}{3}+\frac{1}{4}+\frac{2}{15}+\frac{1}{12}+\frac{2}{35}\cdots+\frac{2}{(2017)(2019)}=$$
$$\frac{2}{1\cdot 3}+\frac{2}{2\cdot 4}+\frac{2}{3\cdot 5}+\frac{2}{4\cdot 6}+\frac{2}{5\cdot 7}\cdots+\frac{2}{(2017)(2019)}=$$
$$\frac{2}{1(1+2)}+\frac{2}{2(2+2)}+\frac{2}{3(3+2)}+\frac{2}{4(4+2)}+\frac{2}{5(5+2)}\cdots+\frac{2}{2017(2017+2)}$$

   Now $\frac{2}{n(n+2)}$ is the same as $\frac{1}{n}-\frac{1}{n+2}$, so the sum above can be rewritten as:

$$1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+\frac{1}{5}-\frac{1}{7}+\cdots+\frac{1}{2015}-\frac{1}{2017}+\frac{1}{2016}-\frac{1}{2018}+\frac{1}{2017}-\frac{1}{2019}$$

   Rearranging we get:

$$1+\frac{1}{2}-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}+\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}+\cdots-\cancel{\frac{1}{2017}}+\cancel{\frac{1}{2017}}-\frac{1}{2018}-\frac{1}{2019}=$$
$$1+\frac{1}{2}-\frac{1}{2018}-\frac{1}{2019}=\frac{3}{2}-\left(\frac{1}{2018}+\frac{1}{2019}\right)=\frac{3}{2}-\frac{2018+2019}{(2018)(2019)}=\frac{3}{2}-\frac{4037}{(2018)(2019)}$$

2. Function f is defined in the following way:

$$f(0)=1$$
$$f(k)=\frac{f(k-1)}{1+f(k-1)} \text{ for } k\geq 1$$

   What is $f(2018)$?

   (a) $\frac{2017}{2018}$ **(b)** $\frac{1}{2019}$ **(c)** $\frac{2019}{2018}$ **(d)** $\frac{2018}{2019}$ **(e)** None of the above

   SOLUTION **(b)**: Look at the first few values:

$$f(1) = \frac{f(0)}{1+f(0)} = \frac{1}{1+1} = \frac{1}{2}$$

$$f(2) = \frac{f(1)}{1+f(1)} = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$$

$$f(3) = \frac{f(2)}{1+f(2)} = \frac{\frac{1}{3}}{1+\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{4}{3}} = \frac{1}{4}$$

Continuing we can see (and can prove using induction) that

$$f(k) = \frac{1}{k+1}$$

So $f(2018) = \frac{1}{2019}$.

3. How many positive divisors does $2^2 \cdot 3^3 \cdot 4^4 \cdot 5^5 \cdot 6^6 \cdot 7^7$ have?

(a) 5040 **(b)** 20160 **(c)** 8160 **(d)** 7200 **(e)** None of the above

SOLUTION **(c)**:

$$2^2 \cdot 3^3 \cdot 4^4 \cdot 5^5 \cdot 6^6 \cdot 7^7 =$$
$$2^2 \cdot 3^3 \cdot (2^2)^4 \cdot 5^5 \cdot (2 \cdot 3)^6 \cdot 7^7 =$$
$$2^2 \cdot 2^8 \cdot 2^6 \cdot 3^3 \cdot 3^6 \cdot 5^5 \cdot 7^7 =$$
$$2^{16} \cdot 3^9 \cdot 5^5 \cdot 7^7$$

All positive divisors of this numbers must have the form $2^a \cdot 3^b \cdot 5^c \cdot 7^d$, where $0 \le a \le 16$, $0 \le b \le 9$, $0 \le c \le 5$ and $0 \le d \le 7$. So there are 17 choices for a, 10 choices for b, 6 choices for c, and 8 choices for d, giving $17 \cdot 10 \cdot 6 \cdot 8 = 8160$ divisors.

4. $\log_7 5 + \log_{49} 3 =$

(a) $\log_7 5\sqrt{3}$ **(b)** $\log_7 45$

(c) $\log_{49} 75$ **(d)** Both (a) and (c) are correct

(e) Both (b) and (c) are correct

SOLUTION **(d)**: Using the change of base formula to convert both logs to base 7 we get:

$$\begin{aligned}\log_7 5 + \frac{\log_7 3}{\log_7 49} &= \log_7 5 + \frac{\log_7 3}{2} \\ &= \log_7 5 + \frac{1}{2}\log_7 3 \\ &= \log_7 5 + \log_7 \sqrt{3} \\ &= \log_7 5\sqrt{3}\end{aligned}$$

Using the change of base formula again we get:

$$\log_7 5\sqrt{3} = \frac{\log_{49} 5\sqrt{3}}{\log_{49} 7} = \frac{\log_{49} 5\sqrt{3}}{\frac{1}{2}} = 2\log_{49} 5\sqrt{3} = \log_{49}\left(5\sqrt{3}\right)^2 = \log_{49} 75$$

5. The solution to the equation $2^{x+3} = 4 \cdot 3^{2x}$ is:

(a) $\frac{3\ln 2}{2\ln 12 - \ln 2}$ **(b)** $\frac{\ln 2}{\ln 4.5}$ **(c)** $\ln\left(\frac{4}{9}\right)$

(d) $3 - \log_2 12$ **(e)** None of the above

SOLUTION **(b)**: First divide both sides by 2^2 to get

$$2^{x+1} = 3^{2x}$$

Then take the natural logarithm of both sides and use properties of logarithms to write the exponents out in front on both sides, then solve for x as below.

$$\begin{aligned}
\ln 2^{x+1} &= \ln 3^{2x} \\
(x+1)\ln 2 &= x\ln 3^2 \\
x\ln 2 + \ln 2 &= x\ln 9 \\
\ln 2 &= x\ln 9 - x\ln 2 \\
\ln 2 &= x(\ln 9 - \ln 2) \\
x &= \frac{\ln 2}{\ln 9 - \ln 2} \\
&= \frac{\ln 2}{\ln \frac{9}{2}} \\
&= \frac{\ln 2}{\ln 4.5}
\end{aligned}$$

6. Two cards are delt from a standard 52 card deck and placed side by side on a table. What is the probability that the first card is a face card (a jack, a queen or a king) and the second card is a king?

(a) $\frac{3}{169}$ **(b)** $\frac{4}{13}$ **(c)** $\frac{4}{221}$ **(d)** $\frac{3}{221}$ **(e)** None of the above

SOLUTION **(e)**: The probability of second card being a king depends on whether the first card was a king. We can split the situation into two cases:

First card is a king The probability of first card is a king is $4/52 = 1/13$. The probability that the second card is a king, given that the first card was a king, is $3/51 = 1/17$. So the probability that both cards are kings is

$$\frac{1}{13} \cdot \frac{1}{17} = \frac{1}{221}$$

First card is a jack or a queen The probability that the first card is a jack or a queen is $8/52 = 2/13$. The probability that the second card is a king, given that the first card was a jack or a queen, is $4/51$. So the probability that the first card is a jack or a queen and the second card is a king is

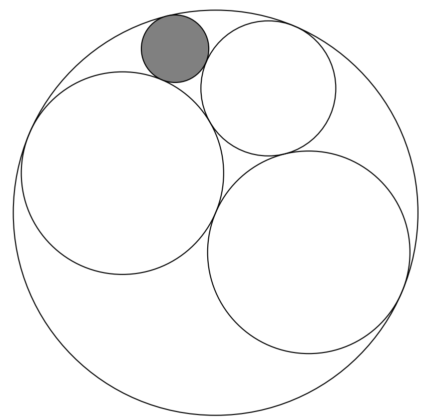
$$\frac{2}{13} \cdot \frac{4}{51} = \frac{8}{663}$$

The two cases are mutually exclusive (the first card cannot be a king and a jack or a queen at the same time), so we can simply add the two probabilities together:

$$\frac{1}{221} + \frac{8}{663} = \frac{11}{663}$$

None of the given answers is correct.

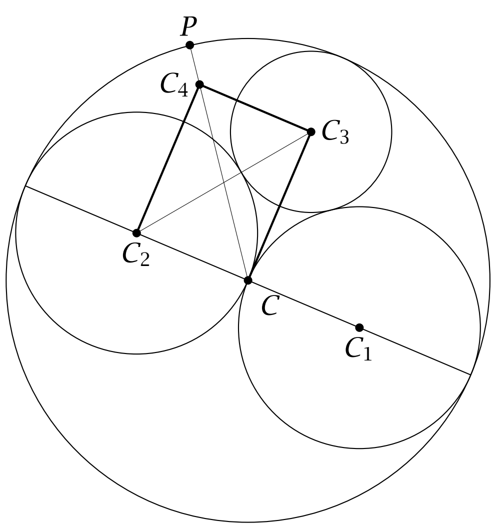
7. A decorative coaster has a design of five circles that are tangent to each other, as shown. The largest circle has radius 2 inches. The two smaller circles each have radius 1 inch. Find the radius of the smallest (filled) circle.



(a) $1/4$

(b) $1/\pi$ **(c)** $10/3$ **(d)** $1/3$ **(e)** None of the above

SOLUTION **(d)**:



The diameters of two 1 inch circles add up to the diameter of the large circle, and since the two circles are tangent to each other, their centers C_1 and C_2 must lie on a common diameter of the large circle, as shown. The center C_3 of the second smallest circle must have an equal distance from C_1 and C_2. It must therefore lie on the perpendicular bisector of C_1C_2. Threfore the angle $\angle C_2CC_3$ is a right angle. Next we will calculate the radius of the circle with center C_3. Call this radius r_3. Since the circle is tangent to the large circle, $|CC_3| + r_3 = 2$. Since the circle with center C_3 is tangent to the circle with center C_2 and with radius 1, the distance of the two centers is $|C_2C_3| = 1 + r_3$. The segment C_2C_3 is the hypotenuse of a right triangle with sides of length $|CC_3|$ and 1, so by Pythagorean Theorem,

$$(1 + r_3)^2 = 1 + (2 - r_3)^2$$

Solve this equation for r_3:

$$
\begin{aligned}
1 + 2r_3 + r_3^2 &= 1 + 4 - 4r_3 + r_3^2 \\
1 + 2r_3 &= 5 - 4r_3 \\
6r_3 &= 4 \\
r_3 &= \frac{2}{3}
\end{aligned}
$$

(An interesting fact: the triangle $\triangle CC_3C_2$ is an example of a *rational right triangle*, a right triangle whose three sides are all rational numbers).
This is where things get interesting: we will show that if we choose C_4 so that the quadrilateral $CC_3C_4C_2$ is a *rectangle*, then the circle with center C_4 and radius 1/3 will be tangent to the circles with centers C_2 and C_3 as well as the large circle, as required:

- Since $|C_3C_4| = 1$, the circle with center at C_4 with radius 1/3 is clearly tangent to the circle with center C_3 and radius 2/3.
- Since $|C_2C_4| = |CC_3| = 2 - r_3 = 4/3$, the circle with center at C_4 and radius 1/3 is tangent to the circle with center C_2 and radius 1.
- Finally, let CP be the radius of the large circle passing through C_4. Then $|C_4P| = 2 - |CC_4| = 2 - |C_2C_3| = 2 - (1 + 2/3) = 1/3$, so the circle centered at C_4 with radius 1/3 is tangent to the large circle.

Therefore the radius of the smallest circle must be 1/3.
The interesting thing here is actually the fact that the four centers C, C_2, C_3 and C_4 form a rectangle.

8. What are the last three digits (the three least significant digits) in 2018^5?

(a) 368 **(b)** 568 **(c)** 768 **(d)** 968 **(e)** None of the above

SOLUTION **(b)**: First of all, if we write 2018^5 as $(2000 + 18)^5$ and expand it, all the terms in the expansion except the last one will have a positive power of 2000, so their three least significant digits will be 0. So the last three digits of 2018^5 will be the same as the last three digits of the last term in the expansion, namely 18^5.
To find the last three digits in 18^5, we use a similar technique: We write 18^5 as $(-2+20)^5$, and expand. All terms in the expansion with factor 20^k for $k \geq 3$ will have the last three digits equal to 0. That will leave the first three terms:

$$(-2)^5 + 5 \cdot (-2)^4 \cdot 20 + \binom{5}{2}(-2)^3 \cdot 20^2$$

However, since $\binom{5}{2} = 10$, this third term will also have the last three digits equal to 0, so the last three digits of 2018^5 are the same as the last three digits of

$$-32 + 5 \cdot 16 \cdot 20 = 1600 - 32 = 1568$$

9. How many positive 5 digit integers can be formed using only the digits 2, 0, 1 and 8 when in each number, each of the digits is used at least once?

(a) 120 **(b)** 180 **(c)** 200 **(d)** 240 **(e)** None of the above

Solution **(b)**: Since we are forming 5 digit integers and there are only 4 digits available, exactly one of them will have to repeat. There are 4 ways to pick the repeating digit. There are now two options:

1. *Zero is not the repeated digit*: In this case there are 4 options for the first digit (zero not allowed), 4 options for the second digit, 3 options for the third, 2 for the fourth and one for the fifth, altogether $4 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 96$ options. This will happen if the repeating digit is 2, 1 or 8.

2. *Zero is the repeated digit*: In this case there are 3 options for the first digit (zero not allowed), 4 options for the second digit, 3 options for the third, 2 for the fourth and one for the fifth, altogether $3 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 72$ options.

However, because of the repeating digit, two such orders will always produce the same integer, so we need to divide the number of orders by 2. Altogether, the number of integers thus produced is

$$\frac{3 \cdot 96 + 72}{2} = \frac{360}{2} = 180$$

10. What is the perimeter of a regular hexagon whose area is $18\sqrt{3}$ square units?

(a) 12 units **(b)** $12\sqrt{2}$ units **(c)** $12\sqrt{3}$ units

(d) $(6\sqrt{3} + 4)$ units **(e)** None of the above

Solution **(c)**: A regular hexagon with side s units consists of 6 congruent equilateral triangles with sides s units. Teh area of each os the triangles is then $3\sqrt{3}$ square units. The area of a triangle is $\frac{1}{2} \cdot \text{base} \cdot \text{height}$ and the height of an equilateral triangle with side s is $\frac{s}{2}\sqrt{3}$, so

$$\begin{aligned} \frac{1}{2} \cdot s \cdot \frac{s}{2}\sqrt{3} &= 3\sqrt{3} \\ \frac{s^2}{4}\sqrt{3} &= 3\sqrt{3} \\ s^2 &= 12 \\ s &= 2\sqrt{3} \end{aligned}$$

The perimeter is then $6s = 6 \cdot 2\sqrt{3} = 12\sqrt{3}$ units.

11. Suppose

$$1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots = 2018$$

and

$$1 + \frac{1}{y} + \frac{1}{y^2} + \frac{1}{y^3} + \cdots = 2019.$$

What is y/x?

(a) $\frac{2019}{2018}$ **(b)** $1 - \frac{1}{2018^2}$ **(c)** $\frac{2017}{2019}$ **(d)** $1 - \frac{1}{2019^2}$

(e) $\frac{2018 \cdot 2019}{2017 \cdot 2020}$

SOLUTION **(b)**: Using the formula for the sum of geometric series,

$$\frac{1}{1 - \frac{1}{x}} = 2018 \text{ so } 1 - \frac{1}{x} = \frac{1}{2018}$$
$$\frac{1}{1 - \frac{1}{y}} = 2019 \text{ so } 1 - \frac{1}{y} = \frac{1}{2019}$$

Then

$$\frac{1}{x} = 1 - \frac{1}{2018} = \frac{2017}{2018}$$
$$\frac{1}{y} = 1 - \frac{1}{2019} = \frac{2018}{2019}$$

and

$$\frac{y}{x} = \frac{\frac{1}{x}}{\frac{1}{y}} = \frac{2017}{2018} \cdot \frac{2019}{2018} = \frac{(2018-1)(2018+1)}{2018^2} = \frac{2018^2 - 1}{2018^2} = 1 - \frac{1}{2018^2}$$

12. Let n be a positive integer such that

$$\frac{n^3 + 6n^2 + 25n + 391}{n + 4}$$

is an integer. How many possible values of n are there?

(a) 0 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** There are infinitely many possibilities for n

SOLUTION **(c)**: Start by dividing the polynomials, for example using synthetic division:

−4	1	6	25	391
		−4	−8	−68
	1	2	17	323

so

$$\frac{n^3 + 6n^2 + 25n + 391}{n+4} = n^2 + 2n + 17 + \frac{323}{n+4}$$

The polynomial part has integer coefficients, so when n is an integer, the polynomial part is an integer as well. In order for the whole rational function to be an integer, the proper fractional part $\frac{323}{n+4}$ must be an integer.
Since $323 = 17 \cdot 19$, the only factors are 1, 17, 19 and 323.
If $n + 4 = 1$, then $n = -3$ which is not a positive integer. If $n + 4 = 17$ or $n + 4 = 19$ or $n + 4 = 323$, we get a positive integer solution n. So there are three such values of n: 13, 15 and 319.

13. When multiplied out,

$$13! = 622\_020800$$

What is the missing digit?

(a) 3 **(b)** 5 **(c)** 7 **(d)** 9 **(e)** None of the above

Solution **(c)**: Since 13! is divisible by 9, if x is the missing digit, $6+2+2+x+0+2+0+8+0+0 = 20 + x$ is divisible by 9. Therefore $x = 7$.

14. Which of the following does *not* have a horizontal asymptote of $y = -1$?

(a) $y = e^{-3x} - 1$ **(b)** $y = \dfrac{3 - \ln x}{2 + \ln x}$ **(c)** $y = \log_2 x - 1$ **(d)** $y = \dfrac{1}{x} - 1$

(e) $y = 3^{-x} - 1$

Solution **(c)**: blabla

15. Which of the following is equal to $\cos\left(\frac{\pi}{12}\right)$?

(a) $\dfrac{\sqrt{2}}{4}$ **(b)** $\dfrac{\sqrt{3}+\sqrt{2}}{4}$ **(c)** $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ **(d)** $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ **(e)** None of the above

Solution **(c)**:

$$\begin{aligned}
\cos\frac{\pi}{12} &= \cos\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)\\
&= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\
&= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}\\
&= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}\\
&= \frac{\sqrt{2}+\sqrt{6}}{4}
\end{aligned}$$

16. Which of the following is the largest?

(a) $\cos\dfrac{\pi}{6}$ **(b)** $\log_2 1$ **(c)** $\log_2 5$ **(d)** $\tan\dfrac{\pi}{4}$ **(e)** $\sqrt{2}$

Solution **(c)**:

▷ $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \approx \frac{1.7}{2} \approx .85$

▷ $\log_2 1 = 0$

▷ $\log_2 5$ is between $\log_2 4 = 2$ and $\log_2 8 = 3$.

▷ $\tan\frac{\pi}{4} = 1$

▷ $\sqrt{2} \approx 1.4$

17. How many ways are there to arange five A's and fourteen B's if each A must be immediately followed by a B?

(a) $\dbinom{19}{5} + \dbinom{19}{4}$ **(b)** $\dbinom{19}{5}$ **(c)** $\dbinom{14}{5} \cdot \dbinom{14}{9}$

(d) $\dbinom{14}{5}$ **(e)** None of the above

Solution **(e)**: Since each A must immediately followed by a B, we can consider the string AB to be a single letter. So we are looking for the number of ways to arrange 5 AB's and the remaining 9 B's, altogether 14 'letters'. Among those 14 'letters', we need to find 5 places for the AB's. So the answer is $\binom{14}{5}$.

18. A circle, an equilateral triangle and a square each have perimeter 12π. Which of the following give the three shapes in ascending order by area?

(a) △, ○, □ **(b)** ○, △, □ **(c)** □, △, ○ **(d)** △, □, ○ **(e)** ○, □, △

Solution **(d)**: The side of an equilateral triangle with perimeter 12π is 4π. The area of the triangle is $\frac{1}{2} \cdot 4\pi \cdot 2\sqrt{3}\pi = 4\sqrt{3}\pi^2$.
The side of a square with perimeter 12π is 3π, so the area of the square is $9\pi^2$.
The radius of a circle with circumference 12π is 6, so the area is 36π.
Now $4\sqrt{3}\pi^2 < 4 \cdot 2\pi^2 < 9\pi^2 < 9 \cdot 4\pi$.

19. A car has wheels with radii 40cm. How many revolutions per minute must a wheel turn so that the car travels 50km/h?

(a) $\dfrac{6520}{\pi}$ **(b)** $\dfrac{3125}{3\pi}$ **(c)** $\dfrac{6520}{3}$ **(d)** $\dfrac{3125}{3}$ **(e)** None of the above

Solution **(b)**:

$$\begin{aligned}\frac{50 \text{ km}}{\text{h}} &= \frac{x \cancel{\text{rev}}}{\cancel{\text{min}}} \cdot \frac{60 \cancel{\text{min}}}{1 \text{ h}} \cdot \frac{2\pi \cdot 40 \cancel{\text{cm}}}{1 \cancel{\text{rev}}} \cdot \frac{1 \cancel{\text{m}}}{100 \cancel{\text{cm}}} \cdot \frac{1 \text{ km}}{1000 \cancel{\text{m}}} \\ &= \frac{x \cdot 60 \cdot 2\pi \cdot 40 \text{ km}}{100 \cdot 1000 \text{ h}} \\ &= \frac{x \cdot 6\pi \text{ km}}{125 \text{ h}}\end{aligned}$$

so

$$x = \frac{125 \cdot 50}{6\pi} = \frac{3125}{3\pi}$$

20. The point (x, y) lies on a circle with radius 3 and center at the origin. Find the maximal value of $x^2 + 3y^2 + 4x$.

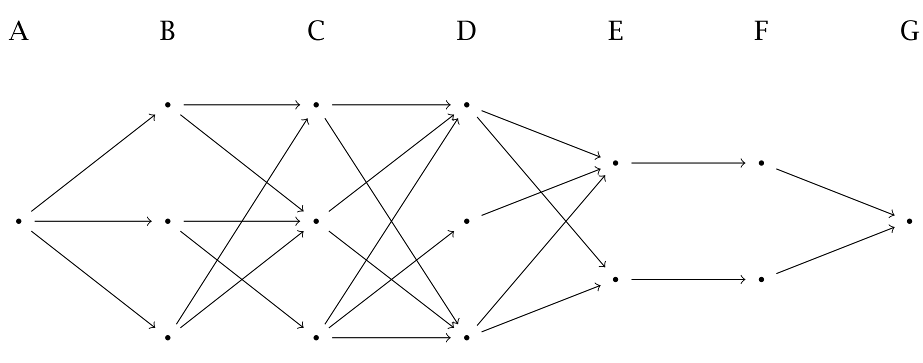
(a) 22 **(b)** 24 **(c)** 36 **(d)** 27 **(e)** 29

Solution **(e)**: We know that $x^2 + y^2 = 9$ and so $y^2 = 9 - x^2$. Then

$$\begin{aligned}x^2 + 3y^2 + 4x &= x^2 + 3(9 - x^2) + 4x \\ &= x^2 + 27 - 3x^2 + 4x \\ &= 27 - 2x^2 + 4x \\ &= 27 - 2(x^2 - 2x) \\ &= 27 - 2(x^2 - 2x + 1 - 1) \\ &= 27 - 2(x^2 - 2x + 1) + 2 \\ &= 29 - 2(x - 1)^2\end{aligned}$$

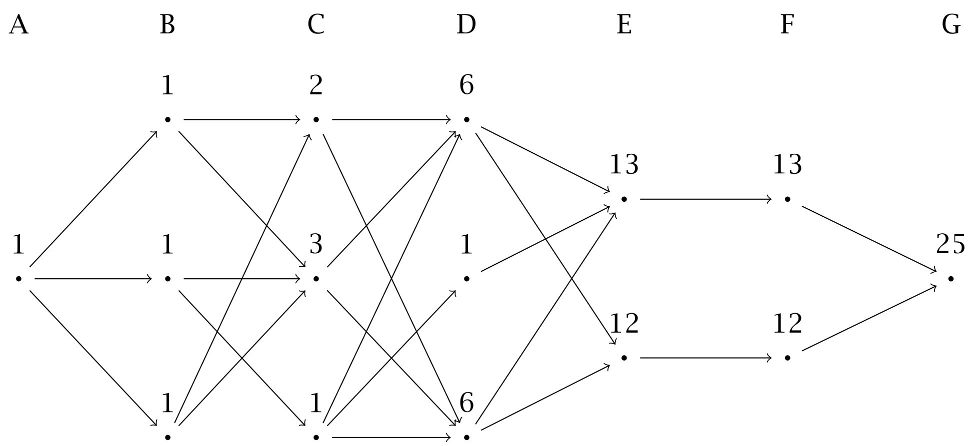
which has maximum value 29.

21. In the diagram, A, B, ..., G refer to successive states through which a traveler must pass in order to get from A to G, moving from left to right. A path consists of a sequence of line segments leading from one state to the next. A path must always move to the next state until reaching state G. Determine the number of possible paths from A to G.



(a) 20 **(b)** 23 **(c)** 24 **(d)** 25 **(e)** 30

Solution **(d)**: We will start at the state A, and going from left to right, for each node in the diagram we will mark in how many different ways can one get to that node from A. For each of the nodes in state B, there is exactly one way to get there. There are two arrows leading to the topmost node in the state C, each coming from a node that has exactly one way of getting to it. So there are $1 + 1 = 2$ different ways of getting to this node. There are $1 + 1 + 1 = 3$ ways to get to the middle node in state C, and only 1 way to get to the lowest node in the state C. There are 3 arrows ending at the topmost node in the state D. The first one arriving from a node which has 2 ways of getting to it, the second from a node that has 3 ways of getting to it, and the last from a node that has only 1 way to get to it. So altogether there are $2 + 3 + 1 = 6$ ways of getting to the topmost node in the state D. Similarly, there is only 1 way to get to the middle node in the state D, and 6 ways to get to the lowest node of the state D. The emerging pattern is: the label for each node will be the sum of the labels at the beginning of all the arrows that end in the given node. So the two nodes in state E will be labeled $6 + 1 + 6 = 13$ and $6 + 6 = 12$. Each of the nodes in state F have only one arrow ending in them, so the labels will also be 13 and 12, respectively. Finally, the last node will have $13 + 12 = 25$ different ways to get to it from the initial node in the state A.



22. How many 10-digit strings of zeros and ones are there that do not contain any consecutive zeros?

(a) 144 **(b)** 512 **(c)** 513 **(d)** 1280 **(e)** None of the above

Solution **(a)**: Let s_n denote the number of n-digit strings of zeros and ones that do not contain any consecutive zeros. Then $s_1 = 2$, $s_2 = 3$. To get a string of length n, you can take a string of length $n - 1$ and append 1, which will give you any string ending with 1, or you can take a string of length $n - 1$ and append 10, which will give you any string ending with 0. So $s_n = s_{n-1} + s_{n-2}$. Then using this recursive formula,

$$s_1 = 2 \quad s_2 = 3 \quad s_3 = 5 \quad s_4 = 8 \quad s_5 = 13$$
$$s_6 = 21 \quad s_7 = 34 \quad s_8 = 55 \quad s_9 = 89 \quad s_{10} = 144$$

23. Find the value of $\sin(2\theta)$ if $\sin\theta + \cos\theta = 0.8$.

(a) $-.36$ **(b)** $-.16$ **(c)** 0 **(d)** $.16$ **(e)** $.36$

SOLUTION **(a)**:

$$\begin{aligned} (\sin\theta + \cos\theta)^2 &= .64 \\ \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta &= .64 \\ 1 + \sin(2\theta) &= .64 \\ \sin(2\theta) &= .64 - 1 = -.36 \end{aligned}$$

24. The numbers x and y satisfy $2^x = 15$ and $15^y = 32$. What is the value of xy?

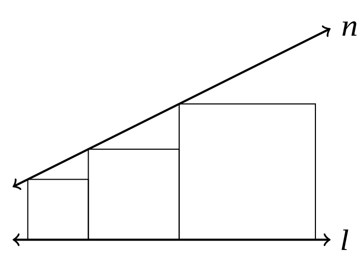
(a) 3 **(b)** 4 **(c)** 5 **(d)** 6 **(e)** None of the above

SOLUTION **(c)**:

$$2^{xy} = (2^x)^y = 15^y = 32$$

so $xy = 5$.

25. Three adjacent squares with increasing side lengths sit on line l as shown, with line n passing through their top left corners. If the two smaller squares have side lengths of 4 and 6, what is the side length of the largest square?



(a) 8

(b) $\frac{26}{3}$ **(c)** 9 **(d)** 10 **(e)** None of the above

SOLUTION **(c)**: The line n has slope $\frac{2}{4} = \frac{1}{2}$, so if the largest square has length x, we have

$$\frac{x-6}{6} = \frac{1}{2}$$

Then $2x - 12 = 6$ so $2x = 18$ and $x = 9$.