Saginaw Valley State University
2018 Math Olympics — Level I Solutions

1. What is the sum of the value(s) of k for which the graph of $y = x^2 + (k-3)x + k$ has exactly one x-intercept?

(a) 10 **(b)** 9 **(c)** 0 **(d)** 3 **(e)** None of the above

SOLUTION **(a)**: The graph will have only one intercept if the equation $x^2 + (k-3)x + k = 0$ has only one solution, which will happen in the case where the expression $y = x^2 + (k-3)x + k$ is a perfect square. This means $x^2 + (k-3)x + k$ has the form $x^2 + 2bx + b^2$. Setting $k = b^2$ and $k - 3 = 2b$, we get $(k-3)^2 = 4k$ or $k^2 - 6k + 9 = 4k$ or $k^2 - 10k + 9 = 0$. Factoring gives us $(k-9)(k-1) = 0$, and according to the zero product property, $k = 9$ or $k = 1$.

The equation $x^2 + (k-3)x + k = 0$ will have only one solution only if the discriminant of the expression is 0. The discriminant for this expression is $b^2 - -4ac$ where $b = k - 3$, $a = 1$ and $c = k$. That gives us $(k-3)^2 - 4k = 0$. This leads to the same equation as the one above.

2. Which of the following is the value of $\frac{2}{3} + \frac{1}{4} + \frac{2}{15} + \frac{1}{12} + \frac{2}{35} + \frac{1}{24} + \frac{2}{63} + \cdots + \frac{2}{(2017)(2019)}$?

(a) $1 - \frac{1}{(2018)(2019)}$ **(b)** $\frac{3}{2} - \frac{4037}{(2018)(2019)}$ **(c)** $1 + \frac{1}{(2018)(2019)}$

(d) $\frac{2018}{2019}$ **(e)** None of the above

SOLUTION **(b)**:

$$\frac{2}{3} + \frac{1}{4} + \frac{2}{15} + \frac{1}{12} + \frac{2}{35} \cdots + \frac{2}{(2017)(2019)} =$$

$$\frac{2}{1\cdot 3} + \frac{2}{2\cdot 4} + \frac{2}{3\cdot 5} + \frac{2}{4\cdot 6} + \frac{2}{5\cdot 7} \cdots + \frac{2}{(2017)(2019)} =$$

$$\frac{2}{1(1+2)} + \frac{2}{2(2+2)} + \frac{2}{3(3+2)} + \frac{2}{4(4+2)} + \frac{2}{5(5+2)} \cdots + \frac{2}{2017(2017+2)}$$

Now $\frac{2}{n(n+2)}$ is the same as $\frac{1}{n} - \frac{1}{n+2}$, so the sum above can be rewritten as:

$$1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{2015} - \frac{1}{2017} + \frac{1}{2016} - \frac{1}{2018} + \frac{1}{2017} - \frac{1}{2019}$$

Rearranging we get:

$$1 + \frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} + \cdots - \cancel{\frac{1}{2017}} + \cancel{\frac{1}{2017}} - \frac{1}{2018} - \frac{1}{2019} =$$

$$1 + \frac{1}{2} - \frac{1}{2018} - \frac{1}{2019} = \frac{3}{2} - \left(\frac{1}{2018} + \frac{1}{2019}\right) = \frac{3}{2} - \frac{2018 + 2019}{(2018)(2019)} = \frac{3}{2} - \frac{4037}{(2018)(2019)}$$

3. Function f is defined in the following way:

$$f(1) = 1$$
$$f(k) = f(k-1) + k \text{ for an integer } k > 1$$

What is $f(4000)$?

(a) 16,004,000 **(b)** 8,002,000 **(c)** 80,000

(d) 7999 **(e)** None of the above

SOLUTION **(b)**: $f(k) = 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$. So $f(4000) = \frac{4000 \cdot 4001}{2}$.

4. Which of the following polynomials is a factor of the polynomial $x^3 + 3x^2y + 3xy^2 + y^3 - 8$?

(a) $x^2 + 2xy + 2x + 2y + y^2 + 4$ **(b)** $(x + y - 2)^2$

(c) $(x + 3y)$ **(d)** $(x + y)^3$

(e) None of the above

SOLUTION **(a)**: Using the Binomial Theorem, the polynomial $x^3 + 3x^2y + 3xy^2 + y^3 - 8$ can be written as $(x + y)^3 - 8 = (x + y)^3 - 2^3$. Using the difference of cubes formula, $(x + y)^3 - 2^3 = [(x + y) - 2][(x + y)^2 + 2(x + y) + 2^2]$. Expanding the second factor gives the expression above.

5. How many positive divisors does $2^2 \cdot 3^3 \cdot 4^4 \cdot 5^5 \cdot 6^6 \cdot 7^7$ have?

(a) 5040 **(b)** 20160 **(c)** 8160 **(d)** 7200 **(e)** None of the above

SOLUTION **(c)**:

$$\begin{aligned} 2^2 \cdot 3^3 \cdot 4^4 \cdot 5^5 \cdot 6^6 \cdot 7^7 &= \\ 2^2 \cdot 3^3 \cdot (2^2)^4 \cdot 5^5 \cdot (2 \cdot 3)^6 \cdot 7^7 &= \\ 2^2 \cdot 2^8 \cdot 2^6 \cdot 3^3 \cdot 3^6 \cdot 5^5 \cdot 7^7 &= \\ 2^{16} \cdot 3^9 \cdot 5^5 \cdot 7^7 \end{aligned}$$

All positive divisors of this numbers must have the form $2^a \cdot 3^b \cdot 5^c \cdot 7^d$, where $0 \le a \le 16$, $0 \le b \le 9$, $0 \le c \le 5$ and $0 \le d \le 7$. So there are 17 choices for a, 10 choices for b, 6 choices for c, and 8 choices for d, giving $17 \cdot 10 \cdot 6 \cdot 8 = 8160$ divisors.

6. A two digit number is written at random. What is the probability that the sum of the digits is 5?

(a) $\frac{5}{89}$ **(b)** $\frac{1}{18}$ **(c)** $\frac{6}{89}$ **(d)** $\frac{1}{15}$ **(e)** $\frac{4}{89}$

SOLUTION **(b)**: There are 90 two digit numbers, and 5 of them have digits that sum to 5: 14, 23, 32, 41 and 50. So the probability is $5/90 = 1/18$.

7. Whole numbers from 1 to 2018 are written in a row. How many times does the digit 8 appear?

(a) 201 **(b)** 601 **(c)** 602 **(d)** 918 **(e)** None of the above

SOLUTION **(c)**:

▷ There is one 8 for each 10 numbers, including the last 8. That makes 202 eights.

▷ There are 10 eights for each hundred numbers. That makes $20 \cdot 10 = 200$ eights.

▷ Finally, there are 100 eights for each 1000 numbers, which makes 200 more eights.
Altogether there are $202 + 200 + 200 = 602$ eights.

8. Find the least common multiple of 375, 175, 168 and 308.

(a) 11550 **(b)** 231000 **(c)** 1155000

(d) 3395700000 **(e)** None of the above

SOLUTION **(b)**: Find the prime factorization of each of the numbers:

▷ $375 = 3 \cdot 5^3$

▷ $175 = 5^2 \cdot 7$

▷ $168 = 2^3 \cdot 3 \cdot 7$

▷ $308 = 2^2 \cdot 7 \cdot 11$

The prime factors of the least common multiple must be 2, 3, 5, 7 and 11, each with the largest power that appears in the prime factorizations of the given numbers. That is

$$2^3 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11$$

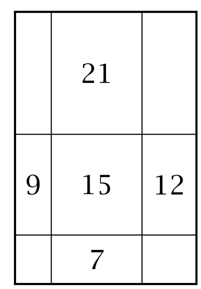
To multiply these, note that $2^3 \cdot 5^3 = 10^3 = 1000$ and $3 \cdot 7 \cdot 11 = 21 \cdot 11 = 210 + 21 = 231$. The least common multiple is 231000

9. How many of the following equations have exactly one integer solution (and possibly any number of non-integer solutions)?
   - ▷ $\sqrt{3x} = 6$
   - ▷ $\sqrt{5x} = 9$
   - ▷ $\sqrt{x} = x$
   - ▷ $\sqrt{x} = 3x$

   (a) None **(b)** 1 **(c)** 2 **(d)** 3 **(e)** All 4

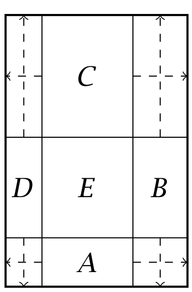
   Solution **(c)**: The first equation is equivalent to $3x = 36$ which has a unique integer solution. The second equation is equivalent to $5x = 81$ which has no integer solutions. The third equation is equivalent to $x = x^2$ which has two integer solutions, 0 and 1. The fourth equation is equivalent to $x = 9x^2$ which has one integer and one non-integer solution. So only the first and the last equation have exactly one integer solution.

10. A large rectangle is subdivided into nine smaller rectangles as shown in the schematic drawing (not to scale). Five of the smaller rectangles have their perimeters written inside them. What is the perimeter of the large rectangle?



   (a) 18 **(b)** 34 **(c)** 42 **(d)** 64 **(e)** None of the above

   Solution **(b)**:



The perimeter of the large rectangle consists of 12 segments, each of which either is, or is the same length as, one of three sides of the small rectangles A, B, C adn D. The fourth sides of the rectangles A, B, C and D form the perimeter of the rectangle E. So the sum of the perimeters of the rectangles A, B, C and D is equal to the sum of the perimeter of the large rectangle and the perimeter of E. The perimeter of the large rectangle is the sum of the perimeters A, B, C and D minus the perimeter E:

$$7 + 12 + 21 + 9 - 15 = 49 - 15 = 34$$

11. Each of 75 students in a science class is a member of at least one of the three clubs: Math club, Chemistry club and Physics club. Exactly 52 of them are in the Math club, exactly 32 of them are in the Chemistry club, and exactly 16 of them are in the Physics club. Exactly 15 of the students are in the Chemistry club only, and exactly 5 of them are in both the Math club and the Physics club, but not the Chemistry club. How many students are in all three of the clubs?

   (a) 3 **(b)** 5 **(c)** 9 **(d)** 15 **(e)** None of the above

SOLUTION **(a)**: There are $75 - 15 = 60$ students who are in the Math club or Physics club. When adding the numbers of members of the Math club and Physics club, we get $52 + 16 = 68$, so there must be 8 students in both the Math and Physics clubs. Out of those, 5 are in Math and Physics clubs only. There muct be $8 - 5 = 3$ students in all three clubs.

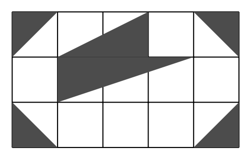
12. How many positive 5 digit integers can be formed when using only the digits 2, 3, 6 and 8 so that in each number, each of the digits is used at least once?

(a) 50 **(b)** 125 **(c)** 200 **(d)** 240 **(e)** None of the above

SOLUTION **(d)**: Since we are forming 5 digit integers and there are only 4 digits available, exactly one of them will have to repeat. There are 4 ways to pick the repeating digit. Now we have 5 digits (with the one repetition), so there are exactly 5! ways to order them. However, because of the repeating digit, two such orders will always produce the same integer, so we need to divide the number of orders by 2. Altogether, the number of integers thus produced is

$$\frac{4 \cdot 5!}{2} = \frac{4 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2} = 240$$

13. A rectangle is subdivided into 15 congruent squares as shown. Each square measures 4 inches on each side. How many square inches are in the unshaded area ?



(a) 7.5 **(b)** 120 **(c)** 10.5 **(d)** 168 **(e)** None of the above

SOLUTION **(d)**: Each square has an area of 4×4, or 16 square inches. The total area of the rectangle is 15×16, or 240 square inches. Each shaded triangle contains half the area of the rectangle that frames it. Each of the four triangles in the corners has half of 16, for a total of 32, while the one in the middle row has half of 3×16, or 24, and the one in the top row right above the middle row one has half of 2×16, or 16. The sum of the shaded areas is $32 + 24 + 16$, or 72. The unshaded area is the difference $240 - 72$, or 168 square inches.

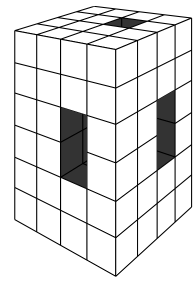
14. If this pattern repeats indefinitely, what is the 2018th letter in the pattern?

SAGINAWSAGINAWSAGINAWSAGINAWSAGINAWSAGINAW...

(a) S **(b)** G **(c)** N **(d)** W **(e)** None of the above

SOLUTION **(e)**: There are 7 letters in the word SAGINAW. To find the 2018th letter, we need to find what is the largest multiple of 7 less than of equal to 2018. Since 2100 is a multiple of 7, and so is 84, we get that $2100 - 84 = 2016$ is a multiple of 7. Therefore $2018 \equiv 2(\pmod 7))$, so we are looking for the second letter in the word SAGINAW, which is A.

15. The figure on the right consists of small cubes. How many of the these cubes are there? Assume that, if there is a "hole" on any surface, then that hole goes all the way through the figure, and that there are no other holes or cavities than the ones shown.



(a) 74 **(b)** 78 **(c)** 89 **(d)** 91 **(e)** None of the above

SOLUTION **(b)**: If the entire figure were uninterrupted by the holes, the total number of cubes would be $4 \times 4 \times 6 = 96$ cubes. The holes intersect in the interior of the figure, so we have to be careful about counting the ones missing. The hole in the left side (2 cubes high) goes all the way to the back, thus eliminating 8 cubes. We are now down to 88 cubes. The hole on the right side (also 2 cubes high) implies another gap of 8 cubes. However, two of them were eliminated from our first count, so there are only 6 more cubes missing. So, we subtract six more cubes to bring our new total to 82 cubes. The hole of a single cube on the top implies a gap of 6 cubes, but two of them were already counted. So, we subtract 4 more cubes which brings us to 78 cubes.

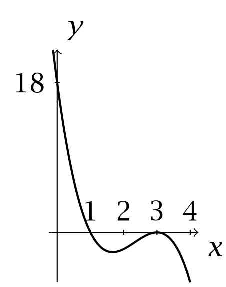
16. An operation $\star$ on real numbers is associative, and $1 \star a = a \star 1 = a$ for any real number a. However, the operation is not commutative, in other words, in general, $a \star b \neq b \star a$. We also know that $7 \star 7 \star 7 = 1$, $5 \star 5 = 1$, and $5 \star 7 = 7 \star 7 \star 5$. Which of the following must be true about $5 \star 7 \star 7$?

(a) $5 \star 7 \star 7 = 7 \star 5$ **(b)** $5 \star 7 \star 7 = 7 \star 7 \star 5$ **(c)** $5 \star 7 \star 7 = 1$

(d) $5 \star 7 \star 7 = 7$ **(e)** None of the above

SOLUTION **(a)**: $5 \star 7 \star 7 = (5 \star 7) \star 7 = (7 \star 7 \star 5) \star 7 = (7 \star 7) \star (5 \star 7) = (7 \star 7) \star (7 \star 7 \star 5) = (7 \star 7 \star 7) \star (7 \star 5) = 1 \star (7 \star 5) = (7 \star 5)$

17. Which of the following expressions is a factored form of the third degree polynomial function $f(x)$ whose graph is given?



(a) $18(x-1)(x-3)^2$

(b) $-2(x-1)(x-3)^2$ **(c)** $18(x-1)^2(x-3)$

(d) $-6(x-1)^2(x-3)$ **(e)** None of the above

SOLUTION **(b)**: The third degree polynomial has a single root at 1 and a double root at 3. Therefore it will be in the form

$$f(x) = k(x-1)(x-3)^2.$$

We also know that $f(0) = k(-1)(-3)^2 = -9k = 18$, which means $k = -2$.

18. If all Martians are green aliens, and some two-headed creatures are Martians, which of the following statements must be true?

    I. Some Martians are not two-headed creatures.

    II. All Martians are two-headed creatures.

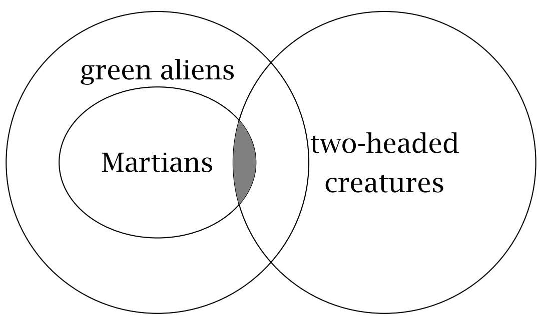
    III. Some green aliens are two-headed creatures.

    (a) I only **(b)** II only

    (c) III only **(d)** I and III only

    (e) None of them must be true

    SOLUTION **(c)**: Draw a Venn diagram for the two statements:



- ▷ The statement "All Martians are green aliens" means that the set of Martians is contained in the set of green aliens.
- ▷ The statement "Some two-headed creatures are Martians" means that there is a non-empty intersection of the set of two-headed creatures and the set of Martians.

The only part of the diagram that is guaranteed to be non-empty is the shaded part. That means that there are some green aliens that also are two-headed creatures, so statement III must be true. However, there is no guarantee that there are any Martians outside of the shaded region, nor that the set of Martians is contained in the set of two-headed creatures.

19. Which of these numbers is largest?

    (a) $\sqrt[3]{3 \cdot \sqrt{7}}$ **(b)** $\sqrt[3]{7 \cdot \sqrt{3}}$ **(c)** $\sqrt{\sqrt[3]{3 \cdot 7}}$ **(d)** $\sqrt{3 \cdot \sqrt[3]{7}}$ **(e)** $\sqrt{7 \cdot \sqrt[3]{3}}$

    SOLUTION **(e)**: Since all the numbers are positive, and sixth power is a strictly increasing function on $[0, \infty)$, raising all the number to the sixth power will preserve their order.

- ▷ $\left(\sqrt[3]{3 \cdot \sqrt{7}}\right)^6 = 3^2 \cdot 7$
- ▷ $\left(\sqrt{3 \cdot \sqrt[3]{7}}\right)^6 = 3^3 \cdot 7$
- ▷ $\left(\sqrt[3]{7 \cdot \sqrt{3}}\right)^6 = 7^2 \cdot 3$
- ▷ $\left(\sqrt{7 \cdot \sqrt[3]{3}}\right)^6 = 7^3 \cdot 3$
- ▷ $\left(\sqrt{\sqrt[3]{3 \cdot 7}}\right)^6 = 3 \cdot 7$

Out of those, $7^3 \cdot 3$ is the largest.

20. What is the sum of the solutions of $x^2 - 5x = 14$?

(a) -5 **(b)** 5 **(c)** 9 **(d)** 19 **(e)** None of the above

Solution **(b)**: Every quadratic equation of the form $x^2 + px + q = 0$ has two solutions, and the sum of the solutions is always $-p$ (while their product is always q).
Or, in this particular case, we can solve the equation by factoring: $x^2 - 5x - 14 = (x - 7)(x + 2)$ which is 0 when $x = 7$ or $x = -2$. Then $7 + (-2) = 5$.

21. A simplified form of the complex fraction $\dfrac{\frac{1}{x} + 1}{-\frac{1}{x} + 1}$ is

(a) $\dfrac{x+1}{1-x}$ **(b)** $\dfrac{x+1}{x-1}$ **(c)** $\dfrac{x-1}{1-x}$ **(d)** $\dfrac{x-1}{x-1}$ **(e)** None of the above

Solution **(b)**: Multiply both the numerator and denominator by x and simplify.

22. Which of the following equations is an equation of a circle with no intercepts and with center that is on the ray that bisects the first quadrant?

(a) $x^2 - 6x + y^2 - 6y = -14$ **(b)** $x^2 - 4x + y^2 - 4y = -4$

(c) $x^2 + 4x + y^2 + 4y = -9$ **(d)** $x^2 - 6x + y^2 - 4y = -12$

(e) None of the above

Solution **(a)**: Start by the process of elimination:

▷ $x^2 - 6x + y^2 - 4y = -2$ does not have center on the line bisecting the first quadrant.

▷ $x^2 + 4x + y^2 + 4y = -9$ has center in the third quadrant, not the first quadrant.

▷ $x^2 - 4x + y^2 - 4y = -4$ has intercepts: setting for example $y = 0$, we get $x^2 - 4x = -4$, which has a solution $x = 2$.

▷ That leaves us with $x^2 - 6x + y^2 - 6y = -14$. If this is a circle at all, it has a center on the right ray. We just need to make sure that it is in fact a circle, and that it has no intercepts. By completing the squares on the left side, we get

$$x^2 - 6x + 9 + y^2 - 6y + 9 = -14 + 18$$
$$(x-3)^2 + (y-3)^2 = 4$$

so it is a circle with center at $(3, 3)$ and radius 2. It satisfies the conditions.

23. What are the last three digits (the three least significant digits) in 2018^5?

(a) 368 **(b)** 568 **(c)** 768 **(d)** 968 **(e)** None of the above

SOLUTION **(b)**: First of all, if we write 2018^5 as $(2000 + 18)^5$ and expand it, all the terms in the expansion except the last one will have a positive power of 2000, so their three least significant digits will be 0. So the last three digits of 2018^5 will be the same as the last three digits of the last term in the expansion, namely 18^5.
To find the last three digits in 18^5, we use a similar technique: We write 18^5 as $(-2+20)^5$, and expand. All terms in the expansion with factor 20^k for $k \geq 3$ will have the last three digits equal to 0. That will leave the first three terms:

$$(-2)^5 + 5 \cdot (-2)^4 \cdot 20 + \binom{5}{2}(-2)^3 \cdot 20^2$$

However, since $\binom{5}{2} = 10$, this third term will also have the last three digits equal to 0, so the last three digits of 2018^5 are the same as the last three digits of

$$-32 + 5 \cdot 16 \cdot 20 = 1600 - 32 = 1568$$

24. What is i^{2018}?

(a) 1 **(b)** i **(c)** -1 **(d)** $-i$ **(e)** None of the above

SOLUTION **(c)**: Since $i^4 = 1$, $i^{2018} = i^{2016+2} = i^{4\cdot 504+2} = (i^4)^{504} \cdot i^2 = 1 \cdot i^2 = -1$.

25. In December, a retailer raised the price of certain product by 25%. In January, the product went on sale so that the discounted price was the same as the original price of the product. How many percent was the discount?

(a) 20% **(b)** 22.5% **(c)** 25% **(d)** 27.5%

(e) It is impossible to determine without knowing the original price.

SOLUTION **(a)**: Let p be the original price. The price in December was $1.25p$. Let x be the percent of the January discount. The discounted price is then $(1 - x/100) \cdot 1.25p$. This is supposed to be equal to p, which means

$$\begin{aligned}
(1 - x/100) \cdot 1.25p &= p \\
(1 - x/100) \cdot 1.25 &= 1 \\
125 - 1.25x &= 100 \\
25 &= 1.25x \\
x &= \frac{25}{1.25} = 20
\end{aligned}$$