Saginaw Valley State University

2017 Math Olympics — Level II Solutions

1. The power set of a set S is the set of all the subsets of S. The empty set is denoted $\varnothing$. Which of the following is a power set of some set?

   (a) $W = \{\{a\}, \{\{b\}\}, \{a, \{b\}\}\}$ **(b)** $X = \{\varnothing, \{a\}, \{b\}, \{a, \{b\}\}\}$

   (c) $Y = \{\varnothing, \{a\}, \{\{b\}\}, \{a, \{b\}\}\}$ **(d)** $Z = \{\varnothing, \{a\}, \{\{b\}\}, \{a, b\}\}$

   (e) $A = \{\varnothing, \{a\}, \{a, b\}\}$

   SOLUTION **(c)**:

   – W is not a power set of any set since $\varnothing \notin W$.

   – X is not a power set of any set since $\{a, \{b\}\}$ is an element of X but $\{\{b\}\}$ is not.

   – Y is the power set of $\{a, \{b\}\}$.

   – Z is not a power set of any set since $\{a, b\}$ is an element of Z but $\{b\}$ is not.

   – A is not a power set of any set since $\{a, b\}$ is an element of A but $\{b\}$ is not.

2. At one of those leadership conferences, there is one of those icebreakers where each of the 50 participants is labeled with a distinct integer between 1 and 50, inclusive. Each odd-numbered participant must shake hands exactly once with each of the other participants, and no even numbered participants may shake hands with each other. How many handshakes take place?

   (a) 450 **(b)** 925 **(c)** 950 **(d)** 1225 **(e)** None of the above

   SOLUTION **(b)**:

   Odd-odd handshakes: Each odd-odd handshake happen between two of the 25 odd numbered participants. Since the order of the two does not matter (a handshake between participants 1 and 3 is also a handshake between participants 3 and 1), the number of odd-odd handshakes is

   $$\binom{25}{2} = 300$$

   Odd-even handshakes: Each of the 25 odd numbered participants must shake hands with each of the 25 even numbered participants. So the number of odd-even handshakes will be

   $$25 \cdot 25 = 25^2 = 625$$

   Since there will be no even-even handshakes, the total number is $300 + 625 = 925$.

3. Find

$$\sum_{k=1}^{2017} i^k$$

where $i^2 = -1$.

(a) -1 **(b)** i **(c)** 1 **(d)** $-i$ **(e)** $1+i$

SOLUTION **(b)**: Note that $i + i^2 + i^3 + i^4 = i + (-1) + (-i) + 1 = 0$ and $i^{n+1} + i^{n+2} + i^{n+3} + y^{n+4} = i^n(i + i^2 + i^3 + i^4) = 0$. Then

$$\begin{aligned}\sum_{k=1}^{2017} i^k &= (i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) \\ &\quad + \ldots + (i^{2013} + i^{2014} + i^{2015} + i^{2016}) + i^{2017} \\ &= 0 + 0 + \cdots + 0 + i^{2017}\end{aligned}$$

and $i^{2017} = i^{2016} \cdot i = \left(i^4\right)^{504} \cdot i = 1^{504} \cdot i = i$.

4. How many integers n with $1000 \le n \le 9999$ have 4 distinct digits in increasing order or decreasing order?

(a) 336 **(b)** 612 **(c)** 720 **(d)** 904 **(e)** None of the above

SOLUTION **(a)**: There is a 1-1 correspondence between such 4-digit integers with increasing digits and the subsets of size 4 of the set $\{1, 2, 3, \ldots, 9\}$. There is a 1-1 correspondence between such 4-digit integers with decreasing digits and the 4 element subsets of $\{0, 1, 2, \ldots, 9\}$. Therefore the number of 4-digit integers with distinct digits in increasing or decreasing order is

$$\binom{9}{4} + \binom{10}{4} = 126 + 210 = 336$$

5. Johnny, Dee Dee, Joey, Tommy, and Marky are in a band. The sums of the ages of each group of four of them are 132, 138, 113, 131, and 126. What is the age of the oldest of the band members?

(a) 39 **(b)** 43 **(c)** 45 **(d)** 47 **(e)** 49

SOLUTION **(d)**: Let S be the sum of all 5 ages. Let x_i, $i = 1, 2, ..., 5$ be the five ages. Let S_i be the sum of all the ages except x_i. Then

$$\begin{aligned} S - x_1 &= 132 \\ S - x_2 &= 138 \\ S - x_3 &= 113 \\ S - x_4 &= 131 \\ S - x_5 &= 126 \\ \hline 5S - S &= 640 \\ 4S &= 640 \\ S &= 160 \end{aligned}$$

Now that we know S, we can find any of the x_is since

$$x_i = S - S_i.$$

We need the largest x_i, so we use the smallest S_i to get

$$x_3 = 160 - 113 = 47.$$

6. For which of the following values of θ is $2^{\cos\theta} > 1$ and $3^{\sin\theta} < 1$?

I. $\frac{5\pi}{6}$ II. $\frac{11\pi}{6}$ III. $\frac{13\pi}{8}$

(a) I only **(b)** II only **(c)** III only **(d)** I and II only

(e) II and III only

SOLUTION **(e)**: If $a > 1$, then $a^x > 1$ for any positive x and $a^x < 1$ for any negative x. Therefore $2^{\cos\theta} > 1$ and $3^{\sin\theta} < 1$, will be true if and only if $\cos\theta > 0$ and $\sin\theta < 0$, which is true exactly when θ is in the fourth quadrant. Out of the three given numbers, the second and third are in the fourth quadrant.

7. $\sin\frac{\pi}{8} + \cos\frac{\pi}{8} =$

(a) $\frac{\sqrt{2}}{4}$ **(b)** $\frac{2+\sqrt{2}}{2}$ **(c)** $\sqrt{\frac{2-\sqrt{2}}{4}}$ **(d)** $\sqrt{1+\frac{1}{\sqrt{2}}}$ **(e)** None of the above

SOLUTION **(d)**: Let $x = \sin\frac{\pi}{8} + \cos\frac{\pi}{8}$ Then

$$x^2 = \sin^2\frac{\pi}{8} + 2\sin\frac{\pi}{8}\cos\frac{\pi}{8} + \cos^2\frac{\pi}{8}$$
$$x^2 = 1 + \sin\frac{\pi}{4}$$
$$x = \sqrt{1+\frac{\sqrt{2}}{2}}$$
$$x = \sqrt{1+\frac{1}{\sqrt{2}}}$$

8. Determine the number of positive divisors of 18,800 that are divisible by 235.

(a) 8 **(b)** 10 **(c)** 12 **(d)** 14 **(e)** 22

SOLUTION **(b)**: Any positive divisor of 18,800 that is divisible by 235 is of the form $235q$ for some positive integer q. Thus, we want to count the number of positive integers q for which $235q$ divides exactly into 18,800. For $235q$ to divide exactly into 18,800, we need $(235q)d = 18{,}800$ for some positive integer d. Simplifying, we want $qd = \frac{18{,}800}{235} = 80$ for some positive integer d. This means that we want to count the positive integers q for which there is a positive integer d such that $qd = 80$. In other words, we want to count the positive divisors of 80. We could do this using prime factorization, or since 80 is relatively small, we can list the divisors: 1, 2, 4, 5, 8, 10, 16, 20, 40, 80. There are 10 such positive divisors, so 18,800 has 10 positive divisors that are divisible by 235.

9. A function f is defined so that if n is an odd integer, then $f(n) = n - 1$ and if n is an even integer, then $f(n) = n^2 - 1$. For example, if $n = 15$, then $f(n) = 14$ and if $n = -6$, then $f(n) = 35$, since 15 is an odd integer and -6 is an even integer. Determine the expression of $f(f(n))$.

(a) $f(f(n)) = n^2 - 2$ if n is even, $f(f(n)) = n^2 - 2n$ if n is odd

(b) $f(f(n)) = n^2 - 2n$ if n is even, $f(f(n)) = n^2 - 2$ if n is odd

(c) $f(f(n)) = 2n^2 - 1$ for all n **(d)** $f(f(n)) = 2n^2 - 2$ for all n **(e)** None of the above

SOLUTION **(a)**: We consider the cases of n even and n odd:
Suppose that n is even.Then n^2 is even and so $f(n) = n^2 - 1$ must be odd. Thus, $f(f(n)) = f(n^2 - 1) = (n^2 - 1) - 1 = n^2 - 2$, since $f(m) = m - 1$ when m is odd.

Suppose that n is odd. Then $f(n) = n - 1$ must be even. Thus, $f(f(n)) = f(n-1) = (n-1)^2 - 1 = n^2 - 2n + 1 - 1 = n^2 - 2n$.

10. If the operation $*$ is defined for all positive real numbers x and y by $x * y = \frac{x+y}{xy}$, which of the following must be true for positive x, y, and z?

I. $x * x = \frac{2}{x}$ II. $x * y = y * x$ III. $x * (y * z) = (x * y) * z$

(a) I only **(b)** I and II only **(c)** I and III only **(d)** II and III only

(e) all three

SOLUTION **(b)**: Let us verify each property:
I. $x * x = \frac{x+x}{x^2} = \frac{2}{x}$;

II. Obviously commutativity is satisfied;

III. A simple example with $x = y = 1$ and $z = 2$ gives us

$$x * (y * z) = 1 * \frac{1+2}{2} = 1 * \frac{3}{2} = \frac{1+\frac{3}{2}}{\frac{3}{2}} = \frac{5}{3}$$
$$(x * y) * z = \frac{1+1}{1} * 2 = 2 * 2 = \frac{2}{2} = 1$$

11. What is the area of the region of the plane determined by the inequality

$$7 \le |x| + |y| \le 13?$$

(a) 169 **(b)** 81 **(c)** 240 **(d)** 120 **(e)** 78

SOLUTION **(c)**: The area is the difference of the areas of two squares with sides $13\sqrt{2}$ and $7\sqrt{2}$, that is, $(13\sqrt{2})^2 - (7\sqrt{2})^2 = 2(169 - 49) = 240$.

12. $2^{7/6} - 2^{2/3} =$

(a) $\sqrt[3]{4}(\sqrt{2}+1)$ **(b)** $\sqrt{2}$ **(c)** $\sqrt{8}(\sqrt{2}-1)$

(d) $\frac{\sqrt[3]{4}}{\sqrt{2}+1}$ **(e)** None of the above

SOLUTION **(d)**:

$$2^{\frac{7}{6}} - 2^{\frac{2}{3}} = 2^{\frac{4}{6}}2^{\frac{3}{6}} - 2^{\frac{2}{3}} = 2^{\frac{2}{3}}\left(2^{\frac{1}{2}} - 1\right) = \sqrt[3]{4}\left(\sqrt{2}-1\right)$$
$$= \frac{\sqrt[3]{4}\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}{\left(\sqrt{2}+1\right)} = \frac{\sqrt[3]{4}\,(2-1)}{\left(\sqrt{2}+1\right)} = \frac{\sqrt[3]{4}}{\sqrt{2}+1}$$

13. If $\ln(\sec\theta - \tan\theta) = x$ then which of the following is true:

(a) $\sec\theta = \frac{e^x+e^{-x}}{2}$ **(b)** $\csc\theta = e^x$ **(c)** $\sec\theta + \tan\theta = e^x - e^{-x}$

(d) $\cos\theta = e^x - e^{-x}$ **(e)** None of the above

SOLUTION **(a)**:

$$\ln(\sec\theta - \tan\theta) = x$$

exponential form:

$$\sec\theta - \tan\theta = e^x$$

add reciprocal to both sides

$$\sec\theta - \tan\theta + \frac{1}{\sec\theta - \tan\theta} = e^x + e^{-x}$$

multiply and divide by conjugate

$$\sec\theta - \tan\theta + \frac{1}{\sec\theta - \tan\theta}\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} = e^x + e^{-x}$$

simplify ($\sec^\theta - \tan^2\theta = 1$)

$$\sec\theta - \tan\theta + \frac{\sec\theta + \tan\theta}{1} = e^x + e^{-x}$$

simplify more

$$2\sec\theta = e^x + e^{-x}$$

divide by 2

$$\sec\theta = \frac{e^x+e^{-x}}{2}$$

14. If $f(x - \frac{1}{2}) = \frac{1}{2}f(x) + 3$ and $f(0) = 0$, what is $f(2)$?

(a) $\frac{45}{8}$ **(b)** $\frac{9}{2}$ **(c)** -90 **(d)** -18 **(e)** None of the above

SOLUTION **(c)**: $0 = f(0) = f(\frac{1}{2} - \frac{1}{2}) = \frac{1}{2}f(\frac{1}{2}) + 3$. Solving for $f(\frac{1}{2})$ gives $f(\frac{1}{2}) = -6$.
Then $-6 = f(\frac{1}{2}) = f(1 - \frac{1}{2}) = \frac{1}{2}f(1) + 3$. Solving for $f(1)$ gives $f(1) = -18$.
Then $-18 = f(1) = f(\frac{3}{2} - \frac{1}{2}) = \frac{1}{2}f(\frac{3}{2}) + 3$. solving for $f(\frac{3}{2})$ gives $f(\frac{3}{2}) = -42$.
Finally, $-42 = f(\frac{3}{2}) = f(2 - \frac{1}{2}) = \frac{1}{2}f(2) + 3$, and solving for $f(2)$ gives $f(2) = -90$.

15. $\cos\left(2\sin^{-1}(x)\right)$ is equivalent to which of the following?

(a) $2\sqrt{1-x^2}$ **(b)** $2x\sqrt{1-x^2}$ **(c)** $\sqrt{1-x^2}$

(d) $1 - 2x^2$, $|x| \le 1$ **(e)** None of the above

SOLUTION **(d)**: Let $\alpha = \sin^{-1}(x)$, which means $x = \sin\alpha$. We want to find $\cos(2\alpha)$.

$$\cos(2\alpha) = 1 - 2\sin^2(\alpha) = 1 - 2x^2$$

16. What condition on c will guarantee that the equation $e^x + ce^{-x} = 2$ has two solutions?

(a) $c < 1$ **(b)** $0 < c < 1$ **(c)** $c < 0$ **(d)** $c > 1$ **(e)** $c > 0$

SOLUTION **(b)**: Multiplying both sides of the equation by e^x and subtracting $2e^x$ from both sides gives us the equation of quadratic type: $(e^x)^2 - 2e^x + c = 0$. Completing the square gives us

$$\begin{aligned}(e^x - 1)^2 &= 1 - c \\ \text{or } e^x &= 1 \pm \sqrt{1 - c}\end{aligned}$$

The right side will have two real values if $1 - c > 0$, or $c < 1$. On the other hand, since e^x must be positive, $1 - \sqrt{1 - 1} > 0$, so $\sqrt{1 - c} < 1$, or $1 - c < 1$, or $c > 0$.

17. Write

$$\frac{(-1)^{2017} + i^{2017}}{5 - \sqrt{-5}}$$

as $a + bi$ where a and b are real numbers.

(a) $\frac{-5-\sqrt{5}}{20} + \frac{5-\sqrt{5}}{20}i$ **(b)** $\frac{-5+\sqrt{5}}{20} - \frac{5-\sqrt{5}}{20}i$ **(c)** $\frac{-5-\sqrt{5}}{30} + \frac{5-\sqrt{5}}{30}i$

(d) $\frac{-5+\sqrt{5}}{30} - \frac{5-\sqrt{5}}{30}i$ **(e)** None of the above

SOLUTION **(c)**: Since $(-1)^{2017} = -1$ and $i^{2017} = i^{2016}i = (i^4)^{504}i = i$, we get

$$\begin{aligned}\frac{(-1)^{2017} + i^{2017}}{5 - \sqrt{-5}} &= \frac{-1 + i}{5 - \sqrt{5}i} = \frac{-1 + i}{5 - \sqrt{5}i} \cdot \frac{5 + \sqrt{5}i}{5 + \sqrt{5}i} \\ &= \frac{(-1 + i)(5 + \sqrt{5}i)}{25 + 5} \\ &= \frac{-5 - \sqrt{5}i + 5i - \sqrt{5}}{30} \\ &= \frac{-5 - \sqrt{5}}{30} + \frac{5 - \sqrt{5}}{30}i\end{aligned}$$

18. If $6\sec^2\theta + 7\tan\theta - 16 = 0$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, what are possible values of $\sin\theta$?

(a) $\frac{5\sqrt{61}}{61}$ and $1/2$ **(b)** $1/3$ and $-\frac{2\sqrt{5}}{5}$ **(c)** $\frac{5\sqrt{61}}{61}$ and $-\frac{2\sqrt{5}}{5}$

(d) $1/3$ and $1/6$ **(e)** None of the above

SOLUTION **(c)**: Let $x = \sin\theta$. Using the Pythagorean identity for sec and tan, rewrite the equation as

$$\begin{aligned} 6(\tan^2\theta + 1) + 7\tan\theta - 16 &= 0 \\ \text{or } 6\tan^2\theta + 7\tan\theta - 10 &= 0 \\ \text{or } (6\tan\theta - 5)(\tan\theta + 2) &= 0 \end{aligned}$$

so $\tan\theta = 5/6$ or $\tan\theta = -2$. Then

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\sin^2\theta}{1-\sin^2\theta} = \frac{x^2}{1-x^2}$$

– If $\tan\theta = 5/6$, we get

$$\begin{aligned} \frac{x^2}{1-x^2} &= \frac{25}{36} \\ 36x^2 &= 25 - 25x^2 \\ 61x^2 &= 25 \\ x &= \pm\frac{5}{\sqrt{61}} = \pm\frac{5\sqrt{61}}{61} \end{aligned}$$

Since $\tan\theta > 0$, θ is in the first quadrant and $\sin\theta > 0$, so $\sin\theta = \frac{5\sqrt{61}}{61}$

– If $\tan\theta = -2$, we get

$$\begin{aligned} \frac{x^2}{1-x^2} &= 4 \\ x^2 &= 4 - 4x^2 \\ 5x^2 &= 4 \\ x &= \pm\frac{2}{\sqrt{5}} = \pm\frac{2\sqrt{5}}{5} \end{aligned}$$

Since $\tan\theta < 0$, θ is in the fourth quadrant and $\sin\theta < 0$, so $\sin\theta = -\frac{2\sqrt{5}}{5}$

19. Find the sum of the integers in the arithmentic sequence

$$11, 28, 45, 62, \ldots, 2017$$

(a) 119,652 **(b)** 119,663 **(c)** 120,655 **(d)** 120,666 **(e)** None of the above

SOLUTION **(d)**: This is an arithmetic sequnce with first term 11 and common difference 17. First we neet to find out how many terms we are adding:

$$\begin{aligned} 11 + 17k &= 2017 \\ 17k &= 2006 \\ k &= 118 \end{aligned}$$

so we need to add 119 terms.

$$\begin{aligned} \sum_{k=0}^{118}(11 + 17k) &= \sum_{k=0}^{118} 11 + 17 \sum_{k=0}^{118} k \\ &= 119 \cdot 11 + 17\left(\frac{118(118+1)}{2}\right) \\ &= 1309 + 119{,}357 \\ &= 120{,}666 \end{aligned}$$

20. Let $f(x) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)x$ for a complex number x. Define $f^{(n)}(x)$ as the n-times composition of f, i.e. $f^{(1)}(x) = f(x)$ and $f^{(n)}(x) = f\left(f^{(n-1)}(x)\right)$. What is $f^{(2017)}(x)$?

(a) $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)x$ **(b)** x **(c)** $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)x$

(d) $-x$ **(e)** None of the above

SOLUTION **(a)**: Since $A = \frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos\left(\pi/3\right) + i\sin\left(\pi/3\right)$ is a sixth root of unity, $A^{2017} = A^{2016} \cdot A = (A^6)^{336} \cdot A = 1^{336} \cdot A = A$.
Since $f^{(1)}(x) = Ax$ and $f^{(n)}(x) = Af^{(n-1)}(x)$, we can see by induction that $f^{(n)}(x) = A^n x$, and so $f^{(2017)}(x) = A^{2017}x = Ax$.

21. In a drawer, Isaac has 10 socks, with two of each of 5 different colors. On Monday, Isaac selects two individual socks at random from the 10 socks in the drawer. On Tuesday, Isaac selects 2 of the remaining 8 socks at random and on Wednesday two of the remaining 6 socks at random. What is the probability that Wednesday is the first day Isaac selects 2 socks of the same color?

(a) $26/315$ **(b)** $8/9$ **(c)** $24/53$ **(d)** $10/189$ **(e)** None of the above

SOLUTION **(a)**: Count backwards: First, choose the color of the two socks you pick on Wednesday in 5 ways. Then there are 4 colors of socks for you to pick two of on Tuesday, but you don't want to two of the same color. Since there are 4 possible colors, the number of ways to do this is $\binom{8}{2} - 4$. Let's call the color of the socks picked on Wednesday Color 1. Then there are 6 socks left, with 2 of, say, Color 2, 2 of Color 3, and 1 each of colors 4 and 5. Since you don't want to pick a two of the same color, the number of ways to do this is $\binom{6}{2} - 2$. Thus the answer is:

$$\frac{5\left(\binom{8}{2} - 4\right)\left(\binom{6}{2} - 2\right)}{\binom{10}{2}\binom{8}{2}\binom{6}{2}} = \frac{5 \cdot 24 \cdot 13}{45 \cdot 28 \cdot 15} = \frac{26}{315}$$

22. Which of the following statements are true?

I. For every real number x, there exists a real number y such that $x + y = 0$.

II. There exists a real number x such that for every real number y, $x + y = 0$.

III. There exists a real number x such that for every real number y, $xy = y$.

(a) I only **(b)** II only **(c)** III only **(d)** I and II only

(e) I and III only

SOLUTION **(e)**:

I. Given a real number x, choose $y = -x$. Then $x + y = x + (-x) = 0$. True.

II. Assume there is such real number x. Choose $y = 0$. Then $x + y = 0$ becomes $x + 0 = 0$, in other words, x must be 0. But if we choose $y = 1$, then $x + y = 0 + 1 \neq 0$, therefore such x cannot exist.

III. Choose $x = 1$. Then for any y, $xy = 1 \cdot y = y$. True.

23. Given that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$, find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}$$

(a) $27^2 + 49^2 + 11$ **(b)** $54 + 98 + 2\sqrt{11}$ **(c)** $7^3 + 11^2 + 5$

(d) $7^3 + 11^7 + 25^3$ **(e)** None of the above

Solution **(c)**: Let's look at the three individual terms:

$$\begin{aligned} a^{(\log_3 7)^2} &= a^{(\log_3 7)(\log_3 7)} \\ &= \left(a^{(\log_3 7)}\right)^{(\log_3 7)} \\ &= 27^{(\log_3 7)} \\ &= (3^3)^{(\log_3 7)} \\ &= (3^{\log_3 7})^3 = 7^3 \end{aligned}$$

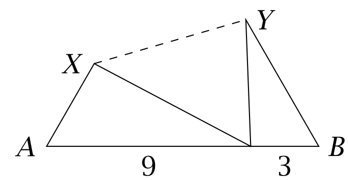
Similarly

$$\begin{aligned} b^{(\log_7 11)^2} &= b^{(\log_7 11)(\log_7 11)} \\ &= \left(b^{(\log_7 11)}\right)^{(\log_7 11)} \\ &= 49^{(\log_7 11)} \\ &= (7^2)^{(\log_7 11)} \\ &= (7^{\log_7 11})^2 = 11^2 \end{aligned}$$

Finally,

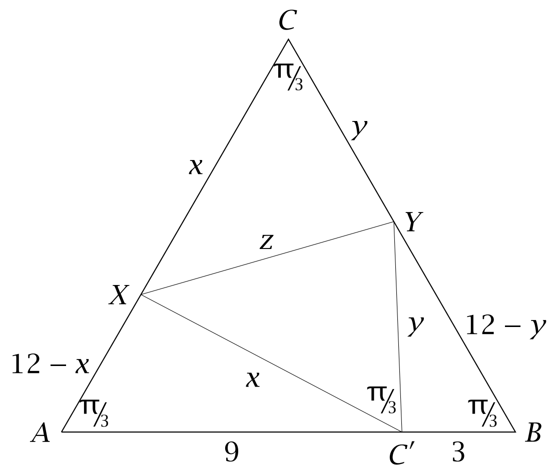
$$\begin{aligned} c^{(\log_{11} 25)^2} &= c^{(\log_{11} 25)(\log_{11} 25)} \\ &= \left(b^{(\log_{11} 25)}\right)^{(\log_{11} 25)} \\ &= \sqrt{11}^{(\log_{11} 25)} \\ &= \left(11^{1/2}\right)^{(\log_{11} 25)} \\ &= (11^{\log_{11} 25})^{1/2} = 5 \end{aligned}$$

24. An equilateral triangle $\triangle ABC$ with side length 12 is folded in such a way that the vertex C touches the side $\overline{AB}$ at a point that is 9 units away from A and 3 units away from B, as shown. Find the length XY.



(a) $\frac{39\sqrt{39}}{35}$ **(b)** $\frac{78\sqrt{6}}{35}$ **(c)** $3\sqrt{7}$ **(d)** $4\sqrt{3}$

(e) None of the above



SOLUTION **(a)**: Label the triangle as shown on the left. The triangles $\triangle XYC$ and $\triangle XYC'$ are congruent. Using the law of cosines on the triangle $\triangle XAC'$ gives us

$$x^2 = 9^2 + (12-x)^2 - 2\cdot 9\cdot(12-x)\cdot\cos\frac{\pi}{3}$$
$$x^2 = 81 + 144 - 24x + x^2 - 2\cdot 9\cdot(12-x)\cdot\frac{1}{2}$$
$$x^2 = 81 + 144 - 24x + x^2 - 108 + 9x$$
$$0 = 117 - 15x$$
$$x = \frac{117}{15} = \frac{39}{5}$$

Doing the same with the triangle $\triangle YBC'$ gives

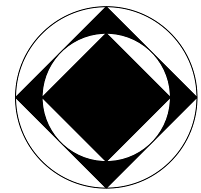
$$y^2 = 3^2 + (12-y)^2 - 2\cdot 3\cdot(12-y)\cdot\cos\frac{\pi}{3}$$
$$y^2 = 9 + 144 - 24y + y^2 - 2\cdot 3\cdot(12-y)\cdot\frac{1}{2}$$
$$y^2 = 9 + 144 - 24y + y^2 - 36 + 3y$$
$$0 = 117 - 21y$$
$$y = \frac{117}{21} = \frac{39}{7}$$

Finally, using the law of cosines on the triangle $\triangle XC'Y$ (or $\triangle XCY$) gives us

$$z^2 = x^2 + y^2 - 2xy\cos\frac{\pi}{3} = \left(\frac{39}{5}\right)^2 + \left(\frac{39}{7}\right)^2 - 2\cdot\left(\frac{39}{5}\right)\left(\frac{39}{7}\right)\frac{1}{2}$$
$$= 39^2\left(\frac{1}{49} + \frac{1}{25} - \frac{1}{35}\right) = 39^2\left(\frac{5^2}{5^2\cdot 7^2} + \frac{7^2}{5^2\cdot 7^2} - \frac{5\cdot 7}{5^2\cdot 7^2}\right)$$
$$= \left(\frac{39}{35}\right)^2(25 + 49 - 35) = \frac{39^3}{35^2}$$

and so $z = \frac{39\sqrt{39}}{35}$.

25. Kenny and Meghen found a cool dart board at a garage sale. It was a white circle in which there was inscribed a black square, in which there was an inscribed white circle, in which there was inscribed a black square, as shown. They were having a debate whether there was more black than white on the dartboard. Meghen said there was more white but Kenny was sure the black area was much larger. He thought that the black area was at least 1.5 times larger than the white area. Rich, who was also shopping at the same garage sale, thought that the black area was larger than the white area, but less than 1.5 times larger. The seller claimed the two areas were equal. Which one of them was right?



(a) Meghen was right, the white area is larger.

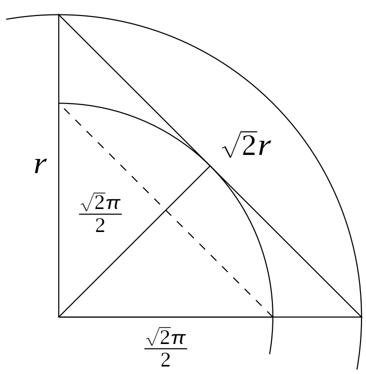
(b) The seller was right, the two areas are equal.

(c) Rich was right, the black area is larger, but less than 1.5 times larger.

(d) Kenny was right, the black area is at least 1.5 times larger than the white area.

(e) It is impossible to determine without knowing the diameter of the large circle.

SOLUTION **(a)**:



Let r be the radius of the large circle. The side of the inscribed square, according to the Pythagorean Theorem, is $\sqrt{r^2+r^2} = \sqrt{2r^2} = \sqrt{2}r$.
The radius of the samller circle, inscribed in the large square is $\frac{\sqrt{2}r}{2}$.
The side of the small square (dashed), inscribed in the small circle is, again using the Pythagorean Theorem,

$$\sqrt{\left(\frac{\sqrt{2}r}{2}\right)^2 + \left(\frac{\sqrt{2}r}{2}\right)^2} = \sqrt{\frac{1}{2}r^2 + \frac{1}{2}r^2} = r$$

The areas of the four regions on the dartboard are as follows:

- Large circle of radius r: area $= \pi r^2$.
- Large square of side $\sqrt{2}r$: area $= 2r^2$.
- Small circle of radius $\frac{\sqrt{2}r}{2}$: area $= \pi r^2/2$.
- Small square of side r: area $= r^2$.

The total white area is then

$$A_w = \pi r^2 - 2r^2 + \frac{\pi}{2}r^2 - r^2 = \left(\frac{3\pi}{2} - 3\right) r^2$$

and the total black area is

$$A_b = 2r^2 - \frac{\pi}{2}r^2 + r^2 = \left(3 - \frac{\pi}{2}\right) r^2.$$

The difference between the two areas is then

$$A_w - A_b = \left(\frac{3\pi}{2} - 3 - 3 + \frac{\pi}{2}\right) r^2 = (2\pi - 6)\, r^2 > 0$$

which shows that the white area is larger.