Saginaw Valley State University
2017 Math Olympics — Level I Solutions

1. What is the last (least significant) digit of 2^{2017}?

   (a) 0 **(b)** 2 **(c)** 4 **(d)** 6 **(e)** 8

   SOLUTION **(b)**: Looking at the first few powers of 2:

$$\begin{array}{ll} 2^1 = 2 & 2^2 = 4 \\ 2^3 = 8 & 2^4 = 16 \\ 2^5 = 32 & 2^6 = 64 \end{array}$$

   and thinking about how a multiplivation by 2 affects the last digit, we discover that the last digt of 2^n is

$$\begin{array}{l} 2 \text{ if } n \equiv 1 \bmod 4 \\ 4 \text{ if } n \equiv 2 \bmod 4 \\ 8 \text{ if } n \equiv 3 \bmod 4 \\ 6 \text{ if } n \equiv 0 \bmod 4 \end{array}$$

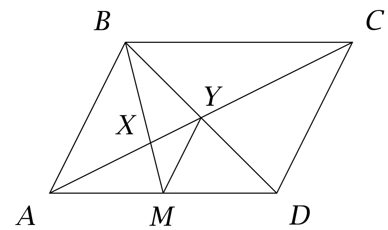
   Since $2017 \equiv 1 \bmod 4$, the last digit is 2.

2. How many integers between 1 and 2017 are divisible by 5 but not 10?

   (a) 200 **(b)** 201 **(c)** 202 **(d)** 403 **(e)** 404

   SOLUTION **(c)**: Since 2016 and 2017 are not divisible by 5, the number for 2017 will be the same as for 2015. Since $2015 = 5 \cdot 403$, the multiples of 5 less than or equal to 2015 are $1 \cdot 5, 2 \cdot 5, 3 \cdot 5, \ldots, 403 \cdot 5$. The even multiples of 5 are also multiples of 10. These are $2 \cdot 5, 4 \cdot 5, 6 \cdot 5, \ldots, 402 \cdot 5$, or $1 \cdot 10, 2 \cdot 10, 3 \cdot 10, \ldots, 201 \cdot 10$. So there are 403 multiples of 5, and 201 of those are also multiples of 10. The answer is $403 - 201 = 202$.

3. Given parallelogram $ABCD$ (as shown) with area of $ABCD = 24$ and diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at point Y. The point M lies on AD so that the area of $\triangle ABM$ is equal to the area of $\triangle MBD$. What is the area of $MYCD$?



(a) 6 **(b)** 8 **(c)** 9 **(d)** 12 **(e)** None of the above

Solution **(c)**: Since the diagonals of a parallelogram bisect each other, the area of $\triangle BYC$ is equal to the area of $\triangle DYC$, as they have congruent bases $\overline{BY}$ and $\overline{DY}$, and shared altitude. For the same reason, the area of $\triangle BYC$ is equal to the area of $\triangle BYA$, which is equal to the area pf $\triangle AYD$. Since these four triangles have equal area and together they completely cover the parallelogram $ABCD$ with no overlap, the area of each of them must be $\frac{24}{4} = 6$.
The triangles $\triangle ABM$ and $\triangle MBD$ have the same area and they share an altitude, so their bases $\overline{AM}$ and $\overline{MD}$ must be congruent. Therefore the area of $\triangle AYM$ is equal to the area of $\triangle MYD$. Since those two triangles completely cover $\triangle AYD$ with no overlap, the area of each of them is $\frac{6}{2} = 3$.
The quadrilateral $MYCD$ consists of the triangle $\triangle DYC$ with area 6 and triangle $\triangle MYD$ with area 3, so the total area is 9.

4. Find

$$\sum_{k=1}^{2017} i^k$$

where $i^2 = -1$.

(a) -1 **(b)** i **(c)** 1 **(d)** $-i$ **(e)** $1+i$

Solution **(b)**: Note that $i + i^2 + i^3 + i^4 = i + (-1) + (-i) + 1 = 0$ and $i^{n+1} + i^{n+2} + i^{n+3} + y^{n+4} = i^n(i + i^2 + i^3 + i^4) = 0$. Then

$$\begin{aligned}\sum_{k=1}^{2017} i^k &= (i + i^2 + i^3 + i^4) \\ &\quad + (i^5 + i^6 + i^7 + i^8) \\ &\quad + \dots \\ &\quad + (i^{2013} + i^{2014} + i^{2015} + i^{2016}) + i^{2017} \\ &= 0 + 0 + \cdots + 0 + i^{2017}\end{aligned}$$

and $i^{2017} = i^{2016} \cdot i = (i^4)^{504} \cdot i = 1^{504} \cdot i = i$.

5. Johnny, Dee Dee, Joey, Tommy, and Marky are in a band. The sums of the ages of each group of four of them are 132, 138, 113, 131, and 126. What is the age of the oldest of the band members?

(a) 39 **(b)** 43 **(c)** 45 **(d)** 47 **(e)** 49

SOLUTION **(d)**: Let S be the sum of all 5 ages. Let x_i, $i = 1, 2, \ldots, 5$ be the five ages. Let S_i be the sum of all the ages except x_i. Then

$$\begin{aligned} S - x_1 &= 132 \\ S - x_2 &= 138 \\ S - x_3 &= 113 \\ S - x_4 &= 131 \\ S - x_5 &= 126 \\ \hline 5S - S &= 640 \\ 4S &= 640 \\ S &= 160 \end{aligned}$$

Now that we know S, we can find any of the x_is since

$$x_i = S - S_i.$$

We need the largest x_i, so we use the smallest S_i to get

$$x_3 = 160 - 113 = 47.$$

6. Determine the number of positive divisors of 18,800 that are divisible by 235.

(a) 8 **(b)** 10 **(c)** 12 **(d)** 14 **(e)** 22

SOLUTION **(b)**: Any positive divisor of 18,800 that is divisible by 235 is of the form $235q$ for some positive integer q. Thus, we want to count the number of positive integers q for which $235q$ divides exactly into 18,800. For $235q$ to divide exactly into 18,800, we need $(235q)d = 18{,}800$ for some positive integer d. Simplifying, we want $qd = \frac{18{,}800}{235} = 80$ for some positive integer d. This means that we want to count the positive integers q for which there is a positive integer d such that $qd = 80$. In other words, we want to count the positive divisors of 80. We could do this using prime factorization, or since 80 is relatively small, we can list the divisors: 1, 2, 4, 5, 8, 10, 16, 20, 40, 80. There are 10 such positive divisors, so 18,800 has 10 positive divisors that are divisible by 235.

7. The sum of the radii of two circles is 10 cm. The circumference of the larger circle is 3 cm greater than the circumference of the smaller circle. Determine the difference between the area of the larger circle and the area of the smaller circle.

(a) π cm^2 **(b)** $\frac{2\pi}{3}$ cm^2 **(c)** 10 cm^2 **(d)** 15 cm^2 **(e)** 20 cm^2

SOLUTION **(d)**: Let the radius of the smaller circle be r cm and let the radius of the larger circle be R cm. Thus, the circumference of the smaller circle is $2\pi r$ cm, the circumference of the larger circle is $2\pi R$ cm, the area of the smaller circle is πr^2 cm^2, and the area of the larger circle is πR^2 cm^2. Since the sum of the radii of the two circles is 10 cm, then $r+R = 10$. Since the circumference of the larger circle is 3 cm larger than the circumference of the smaller circle, then $2\pi R - 2\pi r = 3$, or $2\pi(R - r) = 3$. Then the difference, in cm^2, between the area of the larger circle and the area of the smaller circle is $\pi R^2 - \pi r^2 = \pi(R - r)(R + r) = \frac{1}{2}[2\pi(R - r)](R + r) = \frac{1}{2}(3)(10) = 15$. Therefore, the difference between the areas is 15 cm^2.

8. The sides a, b, and c of a triangle satisfy $\sqrt{a} + \sqrt{b} = \sqrt{c}$. Which of the following describes the triangle?

(a) acute **(b)** scalene **(c)** isosceles

(d) equilateral **(e)** None of the above

SOLUTION **(e)**: Square both sides of $(\sqrt{a} + \sqrt{b})^2 = (\sqrt{c})^2$, to obtain $c = a + b + 2\sqrt{ab}$, which means that c is larger than the sum of a and b. Therefore no such triangle exists.

9. If the operation $*$ is defined for all positive real numbers x and y by $x * y = \frac{x+y}{xy}$, which of the following must be true for positive x, y, and z?

I. $x * x = \frac{2}{x}$

II. $x * y = y * x$

III. $x * (y * z) = (x * y) * z$

(a) I only **(b)** I and II only **(c)** I and III only **(d)** II and III only

(e) all three

SOLUTION **(b)**: Let us verify each property:

I. $x * x = \frac{x+x}{x^2} = \frac{2}{x}$;

II. Obviously commutativity is satisfied;

III. A simple example with $x = y = 1$ and $z = 2$ gives us

$$x * (y * z) = 1 * \frac{1+2}{2} = 1 * \frac{3}{2} = \frac{1+\frac{3}{2}}{\frac{3}{2}} = \frac{5}{3}$$
$$(x * y) * z = \frac{1+1}{1} * 2 = 2 * 2 = \frac{2}{2} = 1$$

10. Two people who work full time and one who works half time are to work together on a project, but their total time allotted to the project is to be the equivalent to one and a half full time work days. If one of the full time workers is budgeted to give half of his work day to the project, and the other is budgeted to give one third of his work day, what part of the half-time worker's day should be given to the project?

    (a) 1/6 **(b)** 1/3 **(c)** 9/5 **(d)** 4/3 **(e)** None of the above

    SOLUTION **(d)**: The three proportions that each person gives to the project should add up to 3/2 of a full time work day. For the two full time workers, the proportion of their work day is the same as the proportion of a normal work day.
    If x is the proportion of the day that the half-time worker gives to the project, then the proportion of a normal work day that he is giving to the project is $(1/2)x$. So the equation to solve is $(1/2) + (1/3) + (1/2)x = 3/2$

11. If four out of five numbers have a sum of 349 and the average of the five numbers is 70, find the fifth number.

    (a) 1 **(b)** -1 **(c)** -337 **(d)** 337 **(e)** None of the above

    SOLUTION **(a)**: Let x be the fifth number $(349 + x)/5 = 70$. Solving for x gives $x = 1$.

12. Elmer drives from Springfield to Lafayette in three hours. Bugs takes a route that is five miles shorter and gets there in half the time. If r is the average rate for Elmer, which of the following is the average rate for Bugs in terms of r?

    (a) $(6r - 10)/3$ **(b)** $(3r - 5)/6$ **(c)** $(r - 5)/2$

    (d) $3r/10$ **(e)** None of the above

    SOLUTION **(a)**: Let d be the distance from Springfield to Lafayette in the route that Elmer took. Then $r = d/3$. In the route that Bugs took, the distance is $d - 5$ and the time is 3/2. So his rate is $(d - 5)/(3/2)$. Solving for d in terms of r gives $d = 3r$. Putting that into Bugs' rate and simplifying by multiplying the numerator and denominator by 2 gives: Bugs' rate

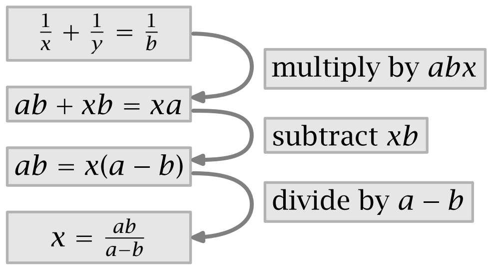
$$\frac{d-5}{3/2} = \frac{3r-5}{3/2} = 2\frac{3r-5}{3} = (6r - 10)/3$$

13. If $a \neq b$ and $1/x + 1/a = 1/b$ then $x =$

(a) $b - a$ **(b)** $1/b - 1/a$ **(c)** $1/a - 1/b$ **(d)** $ab/(a - b)$

(e) None of the above

SOLUTION **(d)**:



14. If $3x^2 - 4x + 5 = 0$ then $(x - 2/3)^2 =$

(a) $-11/9$ **(b)** $-1/3$ **(c)** $19/9$ **(d)** $25/9$ **(e)** None of the above

SOLUTION **(a)**: If $3x^2 - 4x + 5 = 0$ then dividing by 3 gives: $x^2 - \frac{4}{3}x = -\frac{5}{3}$. Adding $\left(\frac{2}{3}\right)^2$ to both sides will complete the square:

$$
\begin{aligned}
x^2 - \frac{4}{3}x + \frac{4}{9} &= -\frac{15}{9} + \frac{4}{9} \\
\text{or } \left(x - \frac{2}{3}\right)^2 &= -\frac{11}{9}
\end{aligned}
$$

15. How many intersection points does the graph of $y = x^2$ have with the circle centered at $(0, 3)$ with radius 2.

(a) none **(b)** 1 **(c)** 2 **(d)** 3 **(e)** 4

SOLUTION **(e)**: The circle centered at $(0, 3)$ with radius 2 has equation $x^2 + (y - 3)^2 = 4$. Substitute $x^2 = y$ into this equation and simplify and you get $y^2 - 7y + 5 = 0$. Using the quadratic formula, this equation has two solutions: $\frac{7+\sqrt{(29)}}{2}$ and $\frac{7-\sqrt{(29)}}{2}$. Since 29 is less than 49, the square root of 29 is less than 7, so both of these solutions are positive. So we can put each of these solutions into the equation $y = x^2$ and for each y we get two solutions for $x = \pm\sqrt{y}$, so this will give four different intersection points.

16. $2^{7/6} - 2^{2/3} =$

(a) $\sqrt[3]{4}(\sqrt{2}+1)$ **(b)** $\sqrt{2}$ **(c)** $\sqrt{8}(\sqrt{2}-1)$

(d) $\frac{\sqrt[3]{4}}{\sqrt{2}+1}$ **(e)** None of the above

SOLUTION **(d)**:

$$2^{\frac{7}{6}} - 2^{\frac{2}{3}} = 2^{\frac{4}{6}}2^{\frac{3}{6}} - 2^{\frac{2}{3}} = 2^{\frac{2}{3}}\left(2^{\frac{1}{2}} - 1\right) = \sqrt[3]{4}\left(\sqrt{2}-1\right)$$
$$= \frac{\sqrt[3]{4}\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}{\left(\sqrt{2}+1\right)} = \frac{\sqrt[3]{4}\,(2-1)}{\left(\sqrt{2}+1\right)} = \frac{\sqrt[3]{4}}{\sqrt{2}+1}$$

17. A collection of centipedes (100 legs), spiders (8 legs) and worms (no legs) has 224 legs and 28 heads, how many worms are there, if there is at least one of each?

(a) 25 **(b)** 23 **(c)** 24 **(d)** 26 **(e)** None of the above

SOLUTION **(b)**: There can be at most 2 centipedes because there are less than 300 legs. If there is only one centipede, that leaves 124 legs for the spiders, but that is not divisible by 8. So there are two centipedes, leaving 24 legs for 3 spiders. $28 - 2 - 3 = 23$ worms.

18. Abby, Benny, and Louie together have 100 pennies. If Abby had six times as many as she has now and Benny had one third as many as he has now, there would still be 100 pennies. Louie says, ``It's not fair, I have fewer than 30.'' How many pennies does Louie have?

(a) 15 **(b)** 25 **(c)** 28 **(d)** 29 **(e)** None of the above

SOLUTION **(a)**: If A is the number of pennies Abby has and B is the number of pennies Benny has, then A and B must satisfy $A + B = 6A + 1/3B$ and $70 < A + B < 100$. If we multiply both sides of the equation by 3 and simplify we get $2B = 15A$. Since A and B are integers, and 2 and 15 don't share any factors, we must have $A = 2k$ and $B = 15k$ for some integer k. So $A + B = 17k$. Since $A + B$ has to be between 70 and 100, k has to be 5. So $A + B = 85$, and Louie has $100 - 85 = 15$ pennies.

19. Find the sum of the integers in the arithmetic sequence

$$11, 28, 45, 62, \ldots, 2017$$

(a) 119,652 **(b)** 119,663 **(c)** 120,655 **(d)** 120,666 **(e)** None of the above

SOLUTION **(d)**: This is an arithmetic sequnce with first term 11 and common difference 17. First we neet to find out how many terms we are adding:

$$\begin{aligned} 11 + 17k &= 2017 \\ 17k &= 2006 \\ k &= 118 \end{aligned}$$

so we need to add 119 terms.

$$\begin{aligned} \sum_{k=0}^{118}(11 + 17k) &= \sum_{k=0}^{118} 11 + 17\sum_{k=0}^{118} k \\ &= 119 \cdot 11 + 17\left(\frac{118(118+1)}{2}\right) \\ &= 1309 + 119{,}357 \\ &= 120{,}666 \end{aligned}$$

20. If the lines $ax + by = 7$ and $ax - by = 9$ intersect at $(1, 2)$, what are a and b, respectively?

(a) 9 and 7 **(b)** $3/2$ and $1/2$ **(c)** 7 and 9 **(d)** 8 and $-1/2$ **(e)** None of the above

SOLUTION **(d)**: The point $(1, 2)$ must be a solution of both of the equations, so

$$\begin{aligned} a + 2b &= 7 \\ a - 2b &= 9 \end{aligned}$$

Adding the two equations together gives $2a = 16$ or $a = 8$. Subtracting the second equation from the first gives $4b = -2$ or $b = -1/2$.

21. If AB is a number with digits (in base 10) A and B, such that when we place a decimal point between the digits we obtain one-third of the original number minus 14, what is AB?

(a) 11 **(b)** 53 **(c)** 60 **(d)** 120 **(e)** None of the above

SOLUTION **(c)**: Let x be the number AB, then $x = 10A + B$, while $A + B/10 = x/3 - 14$. Multiplying this equation by 10 gives us

$$\frac{10x}{3} - 140 = 10A + B$$

Now we have two ways how to express $10A + B$ using x, so we can set them equal:

$$\frac{10}{3}x = 140 = x$$
$$\frac{7}{3}x = 140$$
$$x = 60$$

22. In a certain class, 60% of students have competed in Math Olympics and 35% of students partipated in the Physics Marathon. Exactly 10% of students did both. What percentage have done neither?

(a) 5% **(b)** 10% **(c)** 15% **(d)** 20%

(e) 0%, how could anyone possibly not participate in at least one of these cool events?

SOLUTION **(c)**: First let us find the percent of students who participated in *at least one* of the two competitions. We take the percent of students that participated in Math Olympics, and add the percent of students who participated in the Physics Marathon, however, if we do that, we end up *double counting* those that participated in both, so to fix that, we need to subtract the percent that did both. So

$$60\% + 35\% - 10\% = 85\%$$

of students partipated in at least one of the competitions. The remaining $100\% - 85\% = 15\%$ participated in neither of them (hard as it is to believe).

23. When you divide a number by 210, you get 11. What do you get when you divide the same number by 14?

(a) 11 **(b)** 66 **(c)** 154 **(d)** 165 **(e)** None of the above

SOLUTION **(d)**: Since $210 = 14 \cdot 15$, the mystery number is $(14 \cdot 15) \cdot 11 = 14 \cdot (15 \cdot 11) = 14 \cdot 165$.

24. Math Olympics has decided to branch into the snack industry. It sells two products, Pythagorean Peanut Mix, containing 35% granola by weight, and Gauss Granola Mix, containing 85% granola by weight. A customer wants a special mixture that contains 65% granola by weight. If they want 10 pounds of the mixture, how many pounds of each product is required?

(a) 5 pounds of each

(b) 4 pounds of Pythagorean Peanut Mix and 6 pounds of Gauss Granola Mix

(c) 6 pounds of Pythagorean Peanut Mix and 4 pounds of Gauss Granola Mix

(d) 2 pounds of Pythagorean Peanut Mix and 8 pounds of Gauss Granola Mix

(e) None of the above

SOLUTION **(b)**: Let p be the number of pounds of Pythagorean Peanut Mix and g be the number of pounds of Gauss Granola Mix required to prepare the special mixture. Thw following two equations must be satisfied.
The total amount of mixture must be 10 pounds:

$$p + g = 10$$

Exactly 65% of the 10 pounds must be granola:

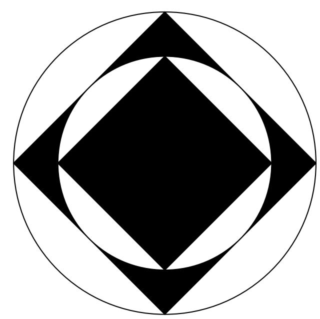
$$.35p + .85g = .65 \cdot 10$$

Multiplying the first equation by 8.5 and the second by -10 gives us

$$\begin{aligned} 8.5p + 8.5g &= 85 \\ -3.5p - 8.5g &= -65 \end{aligned}$$

Adding these equations together results in $5p = 20$, or $p = 4$. The special mix needs 4 pounds of Pythagorean Peanut Mix and 6 pounds of Gauss Granola Mix.

25. Kenny and Meghen found a cool dart board at a garage sale. It was a white circle in which there was inscribed a black square, in which there was an inscribed white circle, in which there was inscribed a black square, as shown. They were having a debate whether there was more black than white on the dartboard. Meghen said there was more white but Kenny was sure the black area was much larger. He thought that the black area was at least 1.5 times larger than the white area. Rich, who was also shopping at the same garage sale, thought that the black area was larger than the white area, but less than 1.5 times larger. The seller claimed the two areas were equal. Which one of them was right?



(a) Meghen was right, the white area is larger.

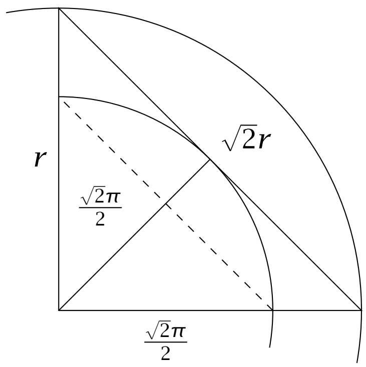
(b) The seller was right, the two areas are equal.

(c) Rich was right, the black area is larger, but less than 1.5 times larger.

(d) Kenny was right, the black area is at least 1.5 times larger than the white area.

(e) It is impossible to determine without knowing the diameter of the large circle.

SOLUTION **(a)**:



Let r be the radius of the large circle. The side of the inscribed square, according to the Pythagorean Theorem, is $\sqrt{r^2+r^2} = \sqrt{2r^2} = \sqrt{2}r$.

The radius of the samller circle, inscribed in the large square is $\frac{\sqrt{2}r}{2}$.

The side of the small square (dashed), inscribed in the small circle is, again using the Pythagorean Theorem,

$$\sqrt{\left(\frac{\sqrt{2}r}{2}\right)^2 + \left(\frac{\sqrt{2}r}{2}\right)^2} = \sqrt{\frac{1}{2}r^2 + \frac{1}{2}r^2} = r$$

The areas of the four regions on the dartboard are as follows:

- Large circle of radius r: area $= \pi r^2$.
- Large square of side $\sqrt{2}r$: area $= 2r^2$.
- Small circle of radius $\frac{\sqrt{2}r}{2}$: area $= \pi r^2/2$.
- Small square of side r: area $= r^2$.

The total white area is then

$$A_w = \pi r^2 - 2r^2 + \frac{\pi}{2}r^2 - r^2 = \left(\frac{3\pi}{2} - 3\right) r^2$$

and the total black area is

$$A_b = 2r^2 - \frac{\pi}{2}r^2 + r^2 = \left(3 - \frac{\pi}{2}\right) r^2.$$

The difference between the two areas is then

$$A_w - A_b = \left(\frac{3\pi}{2} - 3 - 3 + \frac{\pi}{2}\right) r^2 = (2\pi - 6)\, r^2 > 0$$

which shows that the white area is larger.