

Saginaw Valley State University  
2016 Math Olympics — Level II

1. Find the value of the product  $\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{n^2}\right)$ .  
(a)  $\frac{n-1}{n^2}$     (b)  $\frac{n+1}{n^2}$     (c)  $\frac{n+1}{2n}$     (d)  $\frac{n-1}{2n}$     (e) None of the above
  
  2. Let  $x$  and  $y$  be real numbers such that  $x + y = 3$ , and  $x^3 + y^3 = 117$ . What is the value of  $x^2 + y^2$ ?  
(a) 29    (b) 3    (c) 9    (d) 17    (e) 45
  
  3. What is the exact value of  $\frac{\sin(3\alpha) + \sin(\alpha)}{\sin(2\alpha)\cos(\alpha)}$  (for all values of  $\alpha$  such that the above expression is defined)?  
(a)  $-1$     (b) 0    (c) 1    (d) 2    (e) None of the above
  
  4. Three fair dice are rolled. What is the probability that the product of the three outcomes is a prime number? Recall that 1 is not considered to be prime.  
(a) 0    (b)  $\frac{1}{72}$     (c)  $\frac{1}{36}$     (d)  $\frac{1}{24}$     (e) None of the above
  
  5. What is the sum of the digits of all numbers from 1 to 1000?  
(a) 13501    (b) 13601    (c) 13701    (d) 13801    (e) None of the above
  
  6. In order that  $1651817N833536$  equal  $2016^4$ , the letter  $N$  should be replaced by the digit  
(a) 3    (b) 4    (c) 6    (d) 7    (e) 8
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7. A trapezoid has bases  $a$  and  $b$ . Which of the following gives the length of the line segment parallel to both  $a$  and  $b$  that splits the area of the trapezoid in two pieces of ratio 2 : 3?

- (a)  $a + \frac{2}{3}(b - a)$     (b)  $\frac{2a^2+3b^2}{6}$     (c)  $\sqrt{\frac{2a^2+3b^2}{5}}$     (d)  $\frac{2a+3b}{5}$   
(e)  $\sqrt{\frac{2a+3b}{6}}$

8. The lengths of the legs of a right triangle are 15 and 20, respectively. What is the distance from the center of the circle inscribed in the triangle to the altitude of the triangle towards the hypotenuse?

- (a) 0    (b)  $\sqrt{2}$     (c) 2    (d)  $2\sqrt{3}$     (e) None of the above

9. Simplify  $\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} - \frac{1}{2}\cos\alpha}}$  for  $0 \leq \alpha \leq 2\pi$ .

- (a)  $\cos\alpha$     (b)  $\sin\alpha$     (c)  $\cos\left(\frac{\pi}{4} - \frac{\alpha}{4}\right)$   
(d)  $\sin\left(\frac{\alpha}{4}\right)$     (e) None of the above

10.  $\ln(\tan 1^\circ) + \ln(\tan 2^\circ) + \cdots + \ln(\tan 88^\circ) + \ln(\tan 89^\circ) =$

- (a) 0    (b) 1    (c)  $\ln 2$     (d)  $\frac{\sqrt{2}}{2}$     (e) None of the above

11. Find the sum of all positive 2 digit integers that are divisible by each of their digits.

- (a) 495    (b) 546    (c) 582    (d) 630    (e) None of the above

12. Which of the following is not equal to  $\sin^{-1}\left(\sin\frac{46\pi}{7}\right)$ ?

- (a)  $\frac{46\pi}{7}$     (b)  $\frac{3\pi}{7}$   
(c)  $-\tan^{-1}\left(\tan\frac{4\pi}{7}\right)$     (d)  $\cos^{-1}\left(\sqrt{1 - \sin^2\left(\frac{4\pi}{7}\right)}\right)$   
(e) They are all equal to  $\sin^{-1}\left(\sin\frac{46\pi}{7}\right)$
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13. In the expansion of  $\left(x^3 + \frac{y}{x^2}\right)^{15}$ , what is the exponent on  $y$  in the term that has no factors of  $x$ ?

- (a) 5      (b) 6      (c) 9      (d) 10      (e) None of the above

14. When we were preparing this test, we lost the question that went with these answers. Fortunately, since there must be exactly one correct answer for any question, you can decide which one answer below is true.

- (a) The cow jumped over the moon.  
(b) The cow jumped over the moon, or the little dog laughed to see such a sight.  
(c) The little dog laughed to see such a sight, or the dish ran away with the spoon.  
(d) The dish ran away with the spoon and the cow jumped over the moon.  
(e) If the cow didn't jump over the moon, then the little dog laughed to see such a sight.

15.  $\left(\log_{\frac{1}{3}} 9\right) \cdot \left(\log_{27} \frac{1}{5}\right) =$

- (a)  $\frac{1}{2}$   
(b)  $\frac{2}{3}$   
(c) 3  
(d) Since  $\frac{1}{5}$  cannot be the base of a logarithm, this question has no answer.  
(e) None of the above

16. If  $\sin^2 \theta = \frac{1}{3}$ , what is  $72 \cos(2\theta)$ ?

- (a) 0      (b) 24      (c) 72      (d) 124      (e) None of the above

17. Find the number of solutions of the equation

$$\log_{\sqrt{2} \sin x}(1 + \cos x) = 2$$

on the interval  $[0, 2\pi)$ .

- (a) 0      (b) 1      (c) 2      (d) 3      (e) None of the above
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18. Consider the four functions:

$$f(x) = \log_7 x$$

$$g(x) = \log_7(-x)$$

$$h(x) = \log_{\frac{1}{7}} x$$

$$k(x) = \log_{\frac{1}{7}}(-x)$$

$$l(x) = -\log_{\frac{1}{7}} x$$

Which of these functions are increasing on their domain?

- (a)  $f(x)$  only                      (b)  $f(x)$  and  $h(x)$  only                      (c)  $f(x)$  and  $k(x)$  only  
(d)  $f(x)$  and  $l(x)$  only                      (e)  $f(x)$ ,  $k(x)$  and  $l(x)$  only

19. Let  $N = 2^2 3^3 7^4$ . How many factors of  $N^2$  are less than  $N$  but not factors of  $N$ ?

- (a) 72    (b) 75    (c) 97    (d) 98    (e) None of the above

20. Give the exact value of  $\sin \frac{\pi}{10}$ .

- (a)  $\frac{\sqrt{10 + 2\sqrt{5}}}{4}$                       (b)  $\frac{(\sqrt{5} - 1)\sqrt{10 + 2\sqrt{5}}}{8}$                       (c)  $\frac{\sqrt{5} - 1}{4}$   
(d)  $\frac{\sqrt{5} + 1}{4}$                       (e) None of the above

21. An infinite string of digits is formed by writing odd numbers in a row in the following way: you start with one 1, followed by three 3's, then five 5's, seven 7's, then nine 9's, eleven 11's etc. The beginning of the string is 1333555577... What is the 2016th digit in the string?

- (a) 3    (b) 4    (c) 5    (d) 6    (e) None of the above

22. The sum of the solutions of the equation:

$$2(2^x) + 16(2^{-x}) = 33$$

is

- (a) 3                      (b)  $-\frac{1}{3} \log_2(33)$                       (c) 9  
(d) There are no solutions                      (e) None of the above

23. A number  $X$  is chosen at random from the first 100 terms of the arithmetic sequence  $2, 5, 8, \dots$ , and another number  $Y$  is chosen at random from the first 100 terms of the arithmetic sequence  $3, 7, 11, \dots$ . Find  $P(X = Y)$ .

- (a) .0021    (b) .0023    (c) .0025    (d) .0030    (e) .0033

24.  $A$  writes to  $B$  and does not receive an answer. Assuming that one of every  $n$  letters sent is lost in the mail, find the probability that  $B$  received the letter. Assume that  $B$  would have answered the letter if he had received it.

- (a)  $\frac{n}{n-1}$     (b)  $\frac{n-1}{n^2} + \frac{1}{n}$     (c)  $\frac{n-1}{2n-1}$     (d)  $\frac{n}{2n-1}$     (e)  $\frac{n-1}{n^2}$

25. In a survey of customer satisfactions, participants are asked to give a score of 1, 2, 3, or 4 to each of 6 questions. If participants are instructed not to give the same numerical score to more than 4 questions, how many responses are possible?

- (a) 2304    (b) 3840    (c) 4020    (d) 4095    (e) 4096
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