SAGINAW VALLEY STATE UNIVERSITY
2015 MATH OLYMPICS LEVEL II

1. Let $a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ and let $S_n = a_1 + a_2 + a_3 + \ldots + a_n$. Then $S_{99} =$?

(a) 9 **(b)** 10 **(c)** 12 **(d)** 15 **(e)** 21

2. Let f be a function defined on positive integers that satisfies $f(k+2) = 4f(k)$ for all positive integers k. Which of the following could be an equation of f?

(a) $f(k) = 4^k$ **(b)** $f(k) = 4k$ **(c)** $f(k) = \frac{3}{4}(2)^k + \frac{1}{4}(-2)^k$
(d) $f(k) = \frac{1}{2}(4)^k - \frac{1}{2}(-4)^k$ **(e)** none of the above

3. What is the largest prime divisor of $2^{17} - 32$?

(a) 11 **(b)** 13 **(c)** 19 **(d)** 23 **(e)** 29

4. For all angles θ, $\sec^2\theta + \csc^2\theta =$

(a) $\sec^2\theta\csc^2\theta$ **(b)** 1 **(c)** $2 - \tan^2\theta - \cot^2\theta$ **(d)** all of the above
(e) none of the above

5. When a decimal point is placed between the digits of the two-digit integer $n = AB$, where A and B are digits of n, the resulting number is equal to the average of the digits of n. What is the value of $\frac{n}{A}$?

(a) 10.5 **(b)** 11.25 **(c)** $\frac{4}{5}$ **(d)** $\frac{3}{5}$ **(e)** none of the above

6. The product $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) =$?

(a) $\frac{n+1}{n^2}$ **(b)** $\frac{4}{n}$ **(c)** $\frac{n+1}{n}$ **(d)** $\frac{n+1}{2n}$ **(e)** none of the above

7. Let $f(x) = x^2 - 4x$, $x \le 2$. Which of the following gives the correct inverse function for f?

(a) $f^{-1}(x) = -\sqrt{x+4} + 2$ **(b)** $f^{-1}(x) = \sqrt{x+4} + 2$, $x \ge 4$ **(c)** $f^{-1}(x) = \sqrt{x+4}$
(d) $f^{-1}(x) = \frac{x}{x-4}$, $x \le 2$ **(e)** none of the above

8. A bag contains 40 balls, each of which is green or red. Yan reaches into the bag and randomly removes two balls. Each ball in the bag is equally likely to be removed. If the probability that two green balls are removed is $\frac{5}{12}$, how many of the 40 balls are green?

(a) 20 **(b)** 26 **(c)** 32 **(d)** 60 **(e)** none of the above

9. If $f(\frac{x}{1-x}) = \frac{1}{x}$ for all $x \neq 0, 1$, then $f(\tan^2\theta) =$?

(a) $\frac{1}{\tan^2\theta}$ **(b)** $\tan^2\theta$ **(c)** $\csc^2\theta$ **(d)** $\sec^2\theta$ **(e)** none of the above

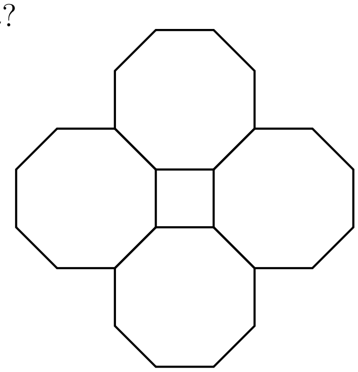
10. $\sin\left(\frac{1}{2}\tan^{-1}(-\frac{3}{4})\right) =$

(a) $\frac{-1}{\sqrt{10}}$ **(b)** $\frac{-3}{\sqrt{74}}$ **(c)** $\frac{-3}{10}$ **(d)** $\frac{1}{\sqrt{2}}$ **(e)** none of the above

11. Let a, b and c be real numbers such that a and c have the same sign, in addition assume that a, b and c are consecutive terms of a geometric sequence (that is, $\frac{c}{b} = \frac{b}{a}$). Then the graph of $y = ax^2 + bx + c$ is

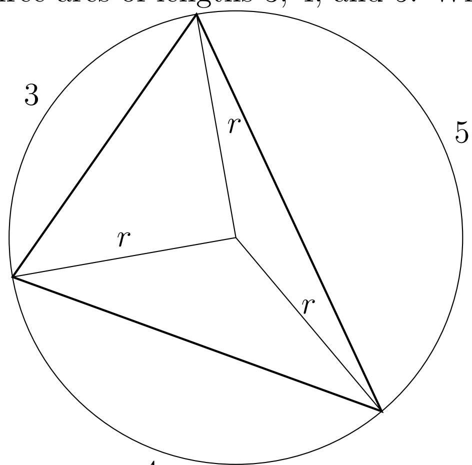
(a) a curve that intersects the x-axis at two distinct points
(b) entirely below or entirely above the x-axis
(c) tangent to the x-axis **(d)** a straight line **(e)** none of the above

12. A regular polygon of m-sides is exactly enclosed (no overlaps, no gaps) by m regular polygons of N sides each. (Shown here for $m = 4$, $n = 8$.) If $m = 10$, what is the value of n?



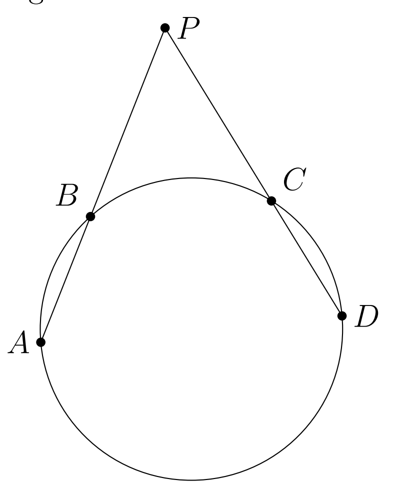
(a) 5 **(b)** 6 **(c)** 14 **(d)** 20 **(e)** 26

13. A triangle is inscribed in one circle. The vertices of the triangle divide the circle into three arcs of lengths 3, 4, and 5. What is the area of the triangle?



(a) 6 **(b)** $\frac{18}{\pi^2}$ **(c)** $\frac{9}{\pi^2}(\sqrt{3}-1)$ **(d)** $\frac{9}{\pi^2}(\sqrt{3}+1)$ **(e)** $\frac{9}{\pi^2}(\sqrt{3}+3)$

14. A circle of radius r has chords AB of length 10 and $CD$of length 7. When AB and CD are extended through B and C, respectively, they intersect at P, which is outside the circle. If angle $\angle APD = 60°$ and $BP = 8$, then $r^2 =$?



(a) 70 **(b)** 71 **(c)** 72 **(d)** 73 **(e)** 74

15. The left most digit of an integer of length 2000 digits is 3. In this integer, any two consecutive digits must be divisible by 17 or 23. The 2000th digit may be either 'a' or 'b'. What is the value of $a + b$?

(a) 3 **(b)** 4 **(c)** 7 **(d)** 10 **(e)** 17

16. If a is chosen randomly from the set $\{1, 2, 3, 4, 5\}$ and b is chosen randomly from the set $\{6, 7, 8\}$, what is the probability that a^b is an even number?

(a) $\frac{1}{3}$ **(b)** $\frac{1}{5}$ **(c)** $\frac{2}{5}$ **(d)** $\frac{3}{5}$ **(e)** none of the above

17. Let $a, b, c, d, \ldots, z$ be twenty six positive real numbers. $\log_a b \cdot \log_b c \cdot \log_c d \cdots \log_y z \log_z(a)$ is equal to which of the following?

(a) $\log_a z$ **(b)** $\log_z(b + c + d + \ldots + z + a)$ **(c)** 1 **(d)** 0
(e) none of the above

18. The product of the digits of a four-digit number is 810. If none of the digits is repeated, the sum of the digits is

(a) 18 **(b)** 19 **(c)** 23 **(d)** 25 **(e)** 22

19. The sum of all values of x that satisfy the equation $(x^2 - 5x + 5)^{x^2+4x-60} = 1$ is

(a) -4 **(b)** 3 **(c)** 1 **(d)** 5 **(e)** 6

20. Let $f(x) = \frac{x-1}{x+1}$ and let $f^{(n)}(x)$ denote the n-fold composition of f with itself. That is, $f^{(1)}(x) = f(x)$ and $f^{(n)}(x) = f(f^{(n-1)}(x))$. Which of the following is $f^{(2015)}(x)$?

(a) $-\frac{1}{x}$ **(b)** $-\frac{x+1}{x-1}$ **(c)** $\frac{1-x}{x+1}$ **(d)** $\frac{1}{x}$ **(e)** $\frac{x-1}{x+1}$

21. If $(2^x - 4^x)^2 + (2^x + 4^x)^2 = 144$, what is x?

(a) $\frac{1}{4}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{3}{4}$ **(d)** $\frac{5}{4}$ **(e)** $\frac{3}{2}$

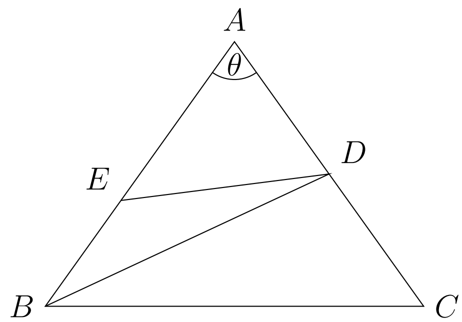
22. The current in a river flows west at a rate of 1.5 ft/sec. A person who rows at a rate of 3 ft/sec in still water is on the south side of the river and wishes to row directly north across the river. In what direction should the person row?

(a) 45° east from north **(b)** 60° east from north **(c)** 30° east from north
(d) 90° east from north **(e)** none of the above

23. The number of positive divisors of 18800 that are divisible by 235 is

(a) 5 **(b)** 10 **(c)** 14 **(d)** 20 **(e)** 24

24. In the figure, $\triangle ABC$ has $AB = AC$ and $\angle BAC < 60°$. Point D is on AC with $BC = BD$. Point E is on AB with $BE = ED$. If $\angle BAC = \theta$, determine $\angle BED$ in terms of θ.



(a) 3θ **(b)** $\frac{\theta}{4}$ **(c)** $\frac{\theta}{2}$ **(d)** θ **(e)** 2θ

25. How many integers can be expressed as a sum of three distinct numbers chosen from the set $\{4, 7, 10, 13, \ldots, 46\}$?

(a) 45 **(b)** 37 **(c)** 36 **(d)** 43 **(e)** 42