SAGINAW VALLEY STATE UNIVERSITY
SOLUTIONS OF 2015 MATH OLYMPICS LEVEL I

1. The value of $\frac{6}{\left(\sqrt{\sqrt{9}-\sqrt{3}}\right)^2}$ is

(a) 1 **(b)** $3-\sqrt{3}$ **(c)** $3+\sqrt{3}$ **(d)** $\frac{6}{3+\sqrt{3}}$ **(e)** $6(3+\sqrt{3})$

The answer is: **(c)**

Solution: We have

$$\frac{6}{\left(\sqrt{\sqrt{9}-\sqrt{3}}\right)^2} = \frac{6}{\left(\sqrt{3-\sqrt{3}}\right)^2} = \frac{1}{3-\sqrt{3}} = \frac{6(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})} = \frac{6(3+\sqrt{3})}{9-3} = 3+\sqrt{3}.$$

2. Suppose that $f(n) = 2f(n+1) - f(n-1)$ for all integers n and $f(1) = 4$ and $f(-1) = 2$. Evaluate $f(2)$.

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** 5

The answer is: **(e)**

Solution: We have $f(0) = 2f(1) - f(-1) = 2(4) - 2 = 6$. Substituting $f(0) = 6$ into $f(1) = 2f(2) - f(0)$ gives $4 = 2f(2) - 6$. Thus $f(2) = 5$.

3. Which of the following equations describes the set of all points (x, y) such that the product of the distance from (x, y) to $(-2, 0)$ and the distance from (x, y) to $(2, 0)$ is 4?

(a) $(x^2 + y^2)^2 = 32$ **(b)** $x^4 + 2x^2y^2 + y^4 + 8y^2 - 8x^2 + 12 = 0$
(c) $x^4 - 8x^2 + y^4 = 0$ **(d)** $(x^2 + y^2)^2 + 8(y^2 - x^2) = 0$
(e) none of the above

The answer is: **(d)**

Solution: Using the distance formula, the product of the distance from (x, y) to $(-2, 0)$ and the distance from (x, y) to $(2, 0)$ is $\sqrt{(x+2)^2 + y^2} \cdot \sqrt{(x-2)^2 + y^2}$. If we set this equal to 4 and square both sides we get the equation: $((x+2)^2+y^2)((x-2)^2+y^2) = 16$. Now we can simplify the left side as follows: $(x^2+4x+4+y^2)(x^2-4x+4+y^2) = ((x^2+y^2+4)+4x)((x^2+y^2+4)-4x) = (x^2+y^2+4)^2 - 16x^2 =$

$(x^2+y^2)^2+8(x^2+y^2)+16-16x^2 = (x^2+y^2)^2+8y^2-8x^2+16$. Subtracting 16 from both sides gives the equation in (d).

4. The average of three positive integers is 28. When two additional integers, s and t, are included, the average of all five integers is 34. What is the average of s and t?

(a) 90 **(b)** 80 **(c)** 70 **(d)** 60 **(e)** none of the above

The answer is: **(e)**

Solution: When the average of three integers is 28, their sum is $3(28) = 84$. When the average of five integers is 34, their sum is $5(34) = 170$. In this case, the difference between the sum of the five integers and the sum of the three integers is $s+t$ which must equal 170-84 = 86. Therefore, $s+t = 86$ and so the average of s and t is $\frac{s+t}{2} = 43$.

5. What is the largest prime divisor of $2^{17} - 32$?

(a) 11 **(b)** 13 **(c)** 19 **(d)** 23 **(e)** 29

The answer is: **(b)**

Solution: $2^{17} - 32 = 2^{17} - 2^5 = 2^5(2^{12} - 1) = 2^5(2^6 - 1)(2^6 + 1) = 2^5 \cdot 63 \cdot 65 = 2^5 \cdot 3^2 \cdot 7 \cdot 5 \cdot 13$

6. What is the smallest positive integer p for which $\sqrt{2^3 \times 5 \times p}$ is an integer?

(a) 1 **(b)** 2 **(c)** 5 **(d)** 10 **(e)** 20

The answer is: **(d)**

Solution: If $\sqrt{2^3 \times 5 \times p}$ is an integer, then $2^3 \times 5 \times p$ is a perfect square. For $2^3 \times 5 \times p$ to be a perfect square, each prime factor must occur an even number of times. For p to be as small as possible, p must have at least one factor of 2 and at least one factor of 5. Thus, the smallest possible value of p is 10.

7. 25 friends are going to order pizza. 7 of the friends like pepperoni on their pizza, 4 of those 7 like both pepperoni and mushrooms, while 3 of those 4 like olives as well. A total of 8 of the friends like mushrooms, and of those 8, 2 of them like mushrooms and olives but no pepperoni. Only 1 person likes only olives. How many of the friends like none of olives, pepperoni, and mushrooms on their pizza?

(a) 13 **(b)** 7 **(c)** 2 **(d)** None of them **(e)** none of the above

The answer is: **(a)**

Solution: We want to find the total who like some kind of topping and subtract from 25. A total of 7 like pepperoni so we can add this to the number that like the other two toppings but not pepperoni. There are a total of 8 that like mushrooms. Subtract off the 4 that like pepperoni with mushroom and there are 4 that like mushrooms without pepperoni. There is 1 that likes olives only. So the total that likes some kind of topping is $7 + 4 + 1 = 12$. So the total that doesn't want any of the three toppings is $25 - 12 = 13$.
Note: you may use a Venn diagram to do this.

8. A bullet is fired at a target at a speed of 100 m/sec. 9 seconds later the sound of the bullet hitting the target comes back to the person firing the shot. If we take the speed of sound to be approximately 350 m/sec, how far away is the target?

(a) 100 meters **(b)** 200 meters **(c)** 700 meters
(d) 900 meters **(e)** none of the above

The answer is: **(c)**

Solution: If we call t the time (in seconds) it takes for the bullet to reach the target, the distance to the target is $100t$ since distance is rate times time. If we call s the time it takes for the sound to travel back from the target, the distance to the target is also $350s$. We also have $t + s = 9$, because the bullet must first hit the target before the sound can travel back, so $s = 9 - t$. This gives the equation $100t = 350(9 - t)$. Solving the equation gives $t = 7$. Plugging $t = 7$ into either one of the distance equations gives 700 meters.

9. If $f(x) = \frac{x^2+1}{x^2-1}$, then $f\left(\frac{1}{x}\right)$ is equal to:

(a) $f(x)$ **(b)** $-f(x)$ **(c)** $-\frac{1}{f(x)}$ **(d)** 1 **(e)** none of the above

The answer is: **(b)**

Solution:

$$f\left(\frac{1}{x}\right) = \frac{\left(\frac{1}{x}\right)^2 + 1}{\left(\frac{1}{x}\right)^2 - 1} = \frac{1 + x^2}{1 - x^2} = -\frac{x^2 + 1}{x^2 - 1}.$$

So $f\left(\frac{1}{x}\right) = -f(x)$.

10. A standard playing card deck has 52 cards made up of 4 different suits with 13 kinds of cards (numbered 1 or "ace" through 10 and three face cards) in each suit. What is the minimum number of cards that must be drawn from a randomly shuffled deck in order to be sure of getting three cards of the same suit?

(a) 9 **(b)** 13 **(c)** 27 **(d)** 42 **(e)** none of the above

The answer is: **(a)**

Solution: In the worst case you could draw 2 cards of each suit, which would be 8 cards, before the 9th one must be the same as 2 of the ones you have already drawn giving you 3 cards in the same suit.

11. Maureen runs a catering service where she gives parties for a flat rate of \$600 for 50 guests or less. For any 5 guests over 50 the rate drops by 50 cents per person. What number of guests will maximize her revenue?

(a) 600 **(b)** 50 **(c)** 60 **(d)** 85 **(e)** none of the above

The answer is: **(d)**

Solution: The revenue is p times x where p is price per unit and x is number of units sold (in this case number of guests). For $x > 50$ we can construct the price function as line with slope -.5/5 with point (50,12) since we know that when $x = 50$ the price is \$12 per person and when price drops by \$.5 dollars for every 5 units of x. So $p - 12 = -.1(x - 50)$ implying $p = 17 - .1x$. So revenue is $R = px = (17 - .1x)x = 17x - .1x^2$. This is a parabola that opens down, so the maximum happens at the vertex which is when $x = -b/2a = -17/-.2 = 85$. When x is less than 50 the revenue is the same as the revenue at the point $x = 50$ on the parabola, so it will not be higher than the revenue at the vertex, so $x = 85$ must be the value that gives the maximum revenue.

12. A chemist has three bottles, each containing a mixture of acid and water: bottle A contains 40 g of which 10% is acid, bottle B contains 50 g of which 20% is acid, and bottle C contains 50 g of which 30% is acid. She uses some of the mixture from each of the bottles to create a mixture with mass 60 g of which 25% is acid. Then she mixes the remaining contents of the bottles to create a new mixture. What percentage of the new mixture is acid?

(a) 20% **(b)** 17.5% **(c)** 15.5% **(d)** 25%
(e) none of the above

The answer is: **(b)**

Solution: Bottle A contains 40 g of which 10% is acid. Thus, it contains $0.1 \times 40 = 4$ g of acid and 40-4 = 36 g of water. Bottle B contains 50 g of which 20% is acid. Thus, it contains $0.2 \times 50 = 10$ g of acid and $50 - 10 = 40$ g of water. Bottle C contains 50 g of which 30% is acid. Thus, it contains $0.3 \times 50 = 15$ g of acid and 50-15 = 35 g of water. In total, the three bottles contain $40 + 50 + 50 = 140$ g, of which $4 + 10 + 15 = 29$ g is acid and 140-29 = 111 g is water. The new mixture has mass 60 g of which 25% is acid. Thus, it contains $0.25 \times 60 = 15$ g of acid and 60-15 = 45 g of water. Since the total mass in the three bottles is initially 140 g and the new mixture has mass 60 g, then the remaining contents have mass 140-60 = 80 g. Since the total mass of acid in the three bottles is initially 29 g and the acid in the new mixture has mass 15 g, then the acid in the remaining contents has mass 29-15 = 14 g. This remaining mixture is thus $\frac{14\text{ g}}{80\text{ g}} \times 100\% = 17.5\%$ acid.

13. Four years ago, Diane was three times as old as Jean was. In five years, Diane will be twice as old as Jean will be. What is the sum of the ages of Diane and Jean?

(a) 25 **(b)** 32 **(c)** 42 **(d)** 44 **(e)** 54

The answer is: **(d)**

Solution: Suppose that Diane's age now is d and Jean's age now is j. Four years ago, Diane's age was $d - 4$ and Jean's age was $j - 4$. In five years, Diane's age will be $d + 5$ and Jean's age will be $j + 5$. From the first piece of given information, $d - 4 = 3(j - 4)$ and so $d - 4 = 3j - 12$ or $d = 3j - 8$. From the second piece of given information, $d + 5 = 2(j + 5)$ and so $d + 5 = 2j + 10$ or $d = 2j + 5$. Equating values of d, we obtain $3j - 8 = 2j + 5$ which gives $j = 13$. Substituting, we obtain $d = 2(13) + 5 = 31$. Therefore, the sum of Diane and Jean ages is 31+13=44.

14. Seven women and seven men attend a party. At this party, each man shakes hands with each other person once. Each woman shakes hands only with men. How many handshakes took place at the party?

(a) 49 **(b)** 70 **(c)** 91 **(d)** 133 **(e)** 182

The answer is: **(b)**

Solution: Seven men can shake hands with each other in $\frac{7\times6}{2} = 21$ ways. The men can shake hands with the women in $7 \times 7 = 49$ ways. Adding these, we get 70 handshakes.

15. Given that $p(x) = (x-1)(x^3+2x-3)+(x-5)(x^3+2x-3)$, what is the sum of the real solutions of $p(x) = 0$?

(a) 4 **(b)** 6 **(c)** -2 **(d)** 7
(e) $p(x) = 0$ has no solution

The answer is: **(a)**

Solution: We have, after factorization, $p(x) = (x^3 + 2x - 3)(x - 1 + x - 5) = (x^3+2x-3)(2x-6) = (x-1)(x^2+x+3)(2x-6)$. (Note that 1 is an obvious root of x^3+2x-3, so if we use long division by $x-1$, we get $x^3+2x-3 = (x-1)(x^2+x+3)$.) Since $x^2 + x + 3$ has no real roots since it has a negative discriminant, thus the sum of roots of $p(x) = 0$ is $3 + 1 = 4$.

16. Given that (x, y) satisfies $x^2 + y^2 = 9$, what is the largest possible value of $x^2 + 3y^2 + 4x$?

(a) 22 **(b)** 24 **(c)** 36 **(d)** 27 **(e)** 29

The answer is: **(e)**

Solution: Replace y^2 with $9 - x^2$ and then complete the square to get $x^2 - 3x^2 + 27 + 4x = 29 - 2x^2 + 4x - 2 = 29 - 2(x - 1)^2$, which is at most 29.

17. Mr. and Mrs. Alpha, Mr. and Mrs. Beta, and Mr. and Mrs. Gamma are standing in a line. How many ways are there line them up so that none of them are standing next to their spouse?

(a) 144 **(b)** 240 **(c)** 288 **(d)** 432 **(e)** none of the above

The answer is: **(b)**

Solution: There are 6 ways to pick the first person in the line. Then there are four ways to pick the next person as it can't be the same as the first person, or be the spouse of that person. Then there are two things that can happen with the third choice. If the third person in line is the spouse of the first person, the fourth person cannot be the spouse of the second person, or that will leave the last couple at the end, so there are two choices for the fourth person, it must be either the husband or wife of the couple that hasn't been used yet, but exactly one choice for each of the fifth and sixth person since the fifth person must be the spouse of the second person. If the third person is not the spouse of the first person and is from the remaining couple, then there are two choices for the fourth person, and in this case there will be two choices for the fifth person. Since there are two ways this can happen there are 8 cases here. So altogether there are 10 different ways to pick the third, fourth, fifth, and sixth places in the line. So there are $6 \times 4 \times 10 = 240$ different ways to line them up.

18. The sides of a right triangle are $a, 2a + 2d$ and $2a + 3d$, with a and d both positive. The ratio of a to d is:

(a) 5:1 **(b)** 27:2 **(c)** 4:1 **(d)** 1:5 **(e)** none of the above

The answer is: **(a)**

Solution: For the right triangle, the longest side is the hypotenuse, therefore $a^2 + (2a+2d)^2 = (2a+3d)^2$. So $a^2 + (4a^2 + 8ad + 4d^2) = 4a^2 + 12ad + 9d^2$ and we have $a^2 - 4ad - 5d^2 = 0$. Factorization yields $(a - 5d)(a + d) = 0$ implying $a = 5d$ or $a = -d$ (dropped since both a and $d > 0$). Hence $a : d = 5d : d = 5 : 1$.

19. Let f be a function defined on positive integers that satisfies $f(k+2) = 4f(k)$. Which of the following could be an equation of f?

(a) $f(k) = 4^k$ **(b)** $f(k) = 4k$ **(c)**$f(k) = \frac{3}{4}(2)^k + \frac{1}{4}(-2)^k$
(d) $f(k) = \frac{1}{2}(4)^k - \frac{1}{2}(-4)^k$ **(e)** none of the above

The answer is: **(c)**

Solution: Note that for (a), (b), and (d) it can be checked that $f(3) \neq 4f(1)$.

If $f(k) = \frac{3}{4}(2)^k + \frac{1}{4}(-2)^k$, then $f(k+2) = \frac{3}{4}(2)^{k+2} + \frac{1}{4}(-2)^{k+2} = \frac{3}{4}(2)^k(2)^2 + \frac{1}{4}(-2)^k(-2)^2 = \frac{3}{4}(2)^k 4 + \frac{1}{4}(-2)^k 4 = 4(\frac{3}{4}(2)^k + \frac{1}{4}(-2)^k) = 4f(k)$.

20. The midpoints of the sides of a triangle are $(1,1), (4,3)$, and $(3,5)$. Find the area of the triangle.

(a) 14 **(b)** 16 **(c)** 18 **(d)** 20 **(e)** 22

The answer is: **(b)**

Solution: The triangle in question has area four times the one obtained by connecting the midpoints with line segments. (Draw the picture.) That triangle has area 4. In fact, you can find the area of any triangle with vertices by augmenting triangular regions to build a rectangle, and then subtracting areas. The rectangle here has vertices $(1,1), (4,1), (4,5)$, and $(1,5)$, so the area of the midpoint triangle is $12 - 3 - 4 - 1 = 4$. Thus the large triangle has area 16.

21. The side lengths of an equilateral triangle and a square are integers. If the triangle and the square have the same perimeter, which of the following is a possible side length of the triangle?

(a) 1 **(b)** 10 **(c)** 18 **(d)** 20 **(e)** 25

The answer is: **(d)**

Solution: Suppose that T is the side length of the equilateral triangle and S is the side length of the square. (Both S and T are integers.) Then, since the perimeters of the triangle and the square are equal, we have $3T = 4S$. Since $3T = 4S$ and each side of the equation is an integer, then T must be divisible by 4 because 4 must divide into $3T$ evenly and it does not divide into 3. The only one of the five possibilities which is divisible by 4 is 20. (We should check that $T = 20$ does indeed yield an integer for S, which it does ($S = 15$).)

22. If $f(x) = ax^2 + bx + c$, and if the y-intercept of f is 1 and the x-intercepts of f are 2 and 3, then $a + b + c =$

(a) 2 **(b)** 6 **(c)** $\frac{1}{3}$ **(d)** $\frac{5}{3}$ **(e)** none of the above

The answer is: **(c)**

Solution 1: Since the x-intercepts of f are 2 and 3, $f(x) = a(x-2)(x-3)$. Since the y-intercept of f is 1, we know $f(0) = 1$, so $a(-2)(-3) = 1$ implying $a = \frac{1}{6}$. Multiplying out $f(x) = \frac{1}{6}(x-2)(x-3) = \frac{1}{6}x^2 - \frac{5}{6}x + 1$. Adding $\frac{1}{6} - \frac{5}{6} + 1 = \frac{1}{3}$.

Solution 2: Alternatively, Since the y-intercept of f is 1, we know $f(0) = 1$, so $c = 1$. Now the x-intercepts of f are 2 and 3, so $f(2) = f(3) = 0$ leading to the system $4a + 2b + 1 = 0$ and $9a + 3b + 1 = 0$. Solving the system gives $a = \frac{1}{6}$ and $b = -\frac{5}{6}$. Thus $a + b + c = \frac{1}{6} - \frac{5}{6} + 1 = \frac{1}{3}$.

23. In the diagram, the rectangle is divided into nine smaller rectangles. The areas of five of these rectangles are given. Determine the area of the rectangle labelled R.

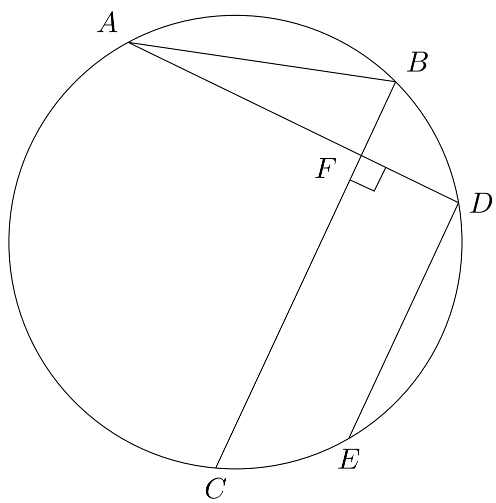
3	1	
	2	R
5		10

(a) 30 **(b)** 20 **(c)** 15 **(d)** 14 **(e)** 12

The answer is: **(e)**

Solution: Let x be the width of the first column.
Since the area of the top left rectangle is 3, the height of the first row is $\frac{3}{x}$.
Since the area of the bottom left rectangle is 5, the height of the third row is $\frac{5}{x}$.
Since the height of the first row is $\frac{3}{x}$ and the area of the top middle rectangle is 1, the width of the middle column is $\frac{x}{3}$.
Thus, the height of the middle row is $\frac{6}{x}$, since the area of the middle rectangle is 2.
Since the height of the third row is $\frac{5}{x}$ and the area of the bottom right rectangle is 10, then the width of the third column is $2x$.
Since the rectangle labelled R has height $\frac{6}{x}$ and width $2x$, then it has area 12.

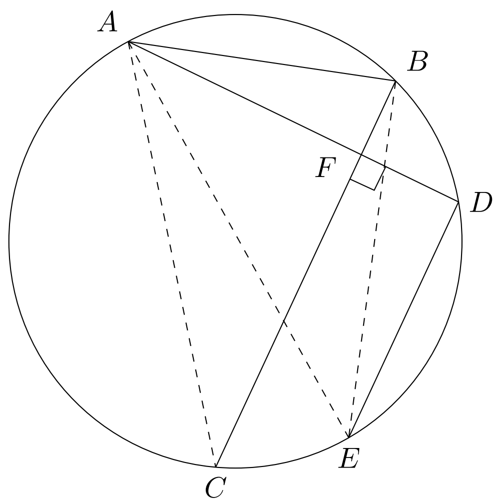
24. In the diagram, AB and BC are chords of the circle with $AB < BC$. If D is the point on the circle such that AD is perpendicular to BC and E is the point on the circle such that DE is parallel to BC, what is $\angle EAC + \angle ABC =$?



(a) 60° **(b)** 75° **(c)** 90° **(d)** 105° **(e)** 120°

The answer is: **(c)**

Solution: Join A to E and C, and B to E.



Since DE is parallel to BC and AD is perpendicular to BC, then AD is perpendicular to DE, i.e., $\angle ADE = 90°$. Therefore, AE is a diameter. Now $\angle EAC = \angle EBC$ since both are subtended by EC. Therefore, $\angle EAC + \angle ABC = \angle EBC + \angle ABC = \angle EBA$ which is indeed equal to 90° as required, since AE is a diameter.

25. For how many integers n, with $2 \leq n \leq 80$, is $\frac{(n-1)(n)(n+1)}{8}$ equal to an integer?
(a) 10 **(b)** 20 **(c)** 59 **(d)** 39 **(e)** 49

The answer is: **(e)**

Solution: If n is an odd integer, then each of $n-1$ and $n+1$ is even. In fact, $n-1$ and $n+1$ are consecutive even integers, so one is a multiple of 4 and the other is divisible is 2 (since it is even). Thus, $(n-1)(n+1)$ contains at least 3 factors of 2, which tells us that $(n-1)(n)(n+1)$ does as well, ie. is divisible by 8. So if n is an odd integer, then $\frac{(n-1)(n)(n+1)}{8}$ is an integer. (There are 39 odd integers between 2 and 80, inclusive.) If n is an even integer, then each of $n-1$ and $n+1$ is odd. Thus, $(n-1)(n)(n+1)$ is divisible by 8 only when n is divisible by 8. (There are 10 multiples of 8 between 2 and 80, inclusive.) Therefore, there are $39 + 10 = 49$ integers n, with $2 \leq n \leq 80$, such that $\frac{(n-1)(n)(n+1)}{8}$ is an integer.

* * * * *

Dear Teachers/Students:

If you do have any suggestions about the competition, or if you have different solutions to any of this year's problems, please send them by mail or e-mail to

Dr. Olivier Heubo-Kwegna
Department of Mathematical Sciences
Sagunaw Valley State University
University Center, MI 46710
oheubokw@svsu.edu

Remember to visit us for information about past competitions at
http://www.svsu.edu/matholympics/

The SVSU Math Olympic Committee would like to express his gratitude to all participants.

* * * * *