SAGINAW VALLEY STATE UNIVERSITY
SOLUTIONS OF 2014 MATH OLYMPICS LEVEL II

1. Define a function f by $f(0) = c$, $f(-101) = -25348$ and $f(n-1) = f(n) + 5n - 1$ for any integer n (for example, $f(-1) = f(0) - 1$). The value of $c =$?

(a) 6 **(b)** 5 **(c)** 4 **(d)** 3 **(e)** 0

The answer is: **(d)**

Solution: We have $f(-1) = f(0) - 1 = c - 1$, $f(-2) = f(-1) - 6 = c - 1 - 6$, $f(-3) = f(-2) - 11 = c - 1 - 6 - 11$, ..., $f(-101) = c - 1 - 6 - 11 - 16 - \ldots - 501 = -25348$.
So $c + \frac{(-1-501)\times 101}{2} = -25348$ and it then follows that $c = 3$.

2. Let $f(x, y)$ be defined by $f(x, 0) = x$ and $f(x, y+1) = f(f(x, y), y)$: which of the following is the largest?

(a) $f(10, 15)$ **(b)** $f(11, 14)$ **(c)**$f(12, 13)$ **(d)** $f(13, 12)$ **(e)** $f(14, 11)$

The answer is: **(e)**

Solution: Remark that $f(x, 1) = f(f(x, 0), 0) = f(x, 0) = x, f(x, 2) = f(f(x, 1), 1) = f(x, 1) = x, \ldots, f(x, n) = x$. So the biggest value is $f(14, 11)$, since it has the largest x value.

3. How many integers can be expressed as a sum of three distinct numbers chosen from the set $\{4, 7, 10, 13, \ldots, 46\}$?

(a) 36 **(b)** 37 **(c)** 42 **(d)** 43 **(e)** 45

The answer is: **(b)**

Solution: Since each number is of the form $1 + 3n, n = 1, 2, 3, \ldots, 15$, the sum of the three numbers will be of the form $3+3k+3l+3m$ where k, l and m are chosen from $\{1, 2, 3, \ldots, 15\}$. So the question is equivalent to the easier question of, "How many distinct integers can be formed by adding three numbers from, $\{1, 2, 3, \ldots, 15\}$?"
The smallest is 1+ 2 + 3 = 6 and the largest is 13 +14 +15 = 42. It is clearly possible to get every sum between 6 and 42 by:
- increasing the sum by one replacing a number with one that is 1 larger or,

- decreasing the sum by one by decreasing one of the addends by 1.
Thus all the integers from 6 to 42 inclusive can be formed. The answer, of course, is 37.

4. How many solutions does the equation

$$\sin(x)\sin(2x)\sin(3x)\cdots\sin(11x)\sin(12x) = 0$$

have in the interval $(0, \pi]$?

(a) 12 **(b)** 24 **(c)** 46 **(d)** 68 **(e)** none of the above

The answer is: **(c)**

Solution: If x is a solution of the equation, then $lx = k\pi$ for some integers k and l with $1 \le l \le 12$. Thus $x = \frac{k}{l}\pi$. Since we want x to be in $(0, \pi]$, the number of solutions equals the number of fractions in $(0, 1]$ with numerator an integer, and denominator a positive integer ≤ 12. Since we want to avoid counting the same solution twice (for example $2\pi/3 = 4\pi/6$), we need to count only fractions in lowest terms. Let $\varphi(n)$ be the number of fractions in lowest terms in $(0, 1]$ with denominator n. Then $\varphi(1) = 1$, $\varphi(2) = 1$, $\varphi(3) = 2$, $\varphi(4) = 2$, $\varphi(5) = 4$, $\varphi(6) = 2$, $\varphi(7) = 6$, $\varphi(8) = 4$, $\varphi(9) = 6$, $\varphi(10) = 4$, $\varphi(11) = 10$, $\varphi(12) = 4$. Adding the above values of φ we get that the number of solutions is 46. (Note that $\varphi(n)$ represents the number of positive integers $\le n$ and relatively prime to n. In elementary number theory class $\varphi(n)$ is the so called Euler function.)

5. Six tickets numbered 1 through 6 are placed in a box. Two tickets are randomly selected and removed together. What is the probability that the smaller of the two numbers on the tickets selected is less than or equal to 4?

(a) $\frac{14}{15}$ **(b)** $\frac{13}{15}$ **(c)** $\frac{4}{5}$ **(d)** $\frac{3}{5}$ **(e)** none of the above

The answer is: **(a)**

Solution 1: The possible pairs of numbers on the tickets are (listed as ordered pairs): $(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6)$, and $(5, 6)$. There are fifteen such pairs. (We treat the pair of tickets numbered 2 and 4 as being the same as the pair numbered 4 and 2.)
The pairs for which the smaller of the two numbers is less than or equal to 4 are: $(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5)$, and $(4, 6)$. There are fourteen such pairs. Therefore, the probability of selecting such a pair of tickets is $\frac{14}{15}$.

Solution 2: We find the probability that the smaller number on the two tickets is NOT less than or equal to 4. Therefore, the smaller number on the two tickets is at least 5. Thus, the pair of numbers must be 5 and 6, since two distinct numbers less than or equal to 6 are being chosen. As in Solution 1, we can determine that there are fifteen possible pairs that we can selected. Therefore, the probability that the smaller number on the two tickets is NOT less than or equal to 4 is $\frac{1}{15}$, so the probability that the smaller number on the two tickets is less than or equal to 4 is $1 - \frac{1}{15} = \frac{14}{15}$.

6. It is known that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. What is the value of $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$?

(a) $\frac{\pi^2}{36}$ **(b)** $\frac{\pi^2}{12}$ **(c)** $\frac{\pi^2}{8}$ **(d)** $\frac{\pi^2}{7}$ **(e)** none of the above

The answer is: **(c)**

Solution: Notice that (separating the odd and even terms)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}.$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \frac{3}{4} \left(\frac{\pi^2}{6} \right) \\
&= \frac{\pi^2}{8}
\end{aligned}$$

7. If $f(x)$ satisfies $2f(x) + f(1-x) = x^2$ for all x, then $f(x) =$

(a) $\frac{x^2-3x+1}{2}$ **(b)** $\frac{x^2+8x-3}{9}$ **(c)** $\frac{x^2+2x-1}{3}$ **(d)** $\frac{4x^2+3x-2}{6}$

(e) none of the above

The answer is: **(c)**

Solution: The given equation $2f(x) + f(1-x) = x^2$ holds for all x. In particular, the equation holds if we replace x with $1-x$. Thus, we deduce that $2f(1-x)+f(1-(1-x)) = (1-x)^2$, i.e., $2f(1-x) + f(x) = (1-x)^2$. Subtracting this equation from twice the given equation results in $3f(x) = 2x^2 - (1-x)^2 = x^2 + 2x - 1$. Thus, $f(x) = \frac{x^2+2x-1}{3}$.

8. If Sam and Peter are among 6 men who are seated at random in a row, the probability that exactly 2 men are seated between them is

(a) 1/10 **(b)** 1/8 **(c)** 1/5 **(d)** 1/4 **(e)** none of the above

The answer is: **(c)**

Solution: The total number of the permutations of 6 is $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$. There are $6 \times 4!$ outcomes with exactly two men seated between Sam and Peter. To see this note that the blanks can be filled in in four factorial ways: S,– ,– , P,– ,– for each of the six possible positions of the S and P. The probability is therefore $6 \cdot 4!/6! = 1/5$.
(Or note that once Peter has been seated, there is just one place out of the five remaining for Sam to sit so that the two are separated by exactly two men.)

9. If $\frac{1}{\cos x} - \tan x = 3$, what is value of $\sin x$?

(a) $-4/5$ **(b)** $4/5$ **(c)** 1 **(d)** 0 **(e)** none of the above

The answer is: **(a)**

Solution: Beginning with the given equation, we have $\frac{1}{\cos x} - \tan x = 3$, i.e., $\frac{1}{\cos x} - \frac{\sin x}{\cos x} = 3$. Multiplying both sides of the equality by $\cos x$ gives $1 - \sin x = 3\cos x$. Now by squaring both sides, we have $1 - 2\sin x + \sin^2 x = 9\cos^2 x$. Replacing $\cos^2 x$ by $1 - \sin^2 x$ and collecting like terms, one obtains $5\sin^2 x - \sin x - 4 = 0$, i.e., $(5\sin x + 4)(\sin x - 1) = 0$. Thus, $\sin x = -4/5$ or $\sin x = 1$.
If $\sin x = 1$, then $\cos x = 0$ which is inadmissible in the original equation. Therefore, $\sin x = -4/5$.

10. Define $f(x) = \sin^6 x + \cos^6 x + k(\sin^4 x + \cos^4 x)$ for some real number k. Then the value of k for which $f(x)$ is constant for all values of x is

(a) 0 **(b)** $\frac{3}{2}$ **(c)** $\frac{-3}{2}$ **(d)** $\frac{5}{2}$ **(e)** $\frac{7}{2}$

The answer is: **(c)**

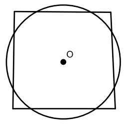
Solution: Since $\sin^2 x + \cos^2 x = 1$, then
$f(x) = (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x) + k(\sin^4 x + \cos^4 x)$
$= (\sin^4 + 2\sin^2 x\cos^2 x + \cos^4 x - 3\sin^2 x\cos^2 x) + k(\sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x - 2\sin^2 x\cos^2 x)$
$= ((\sin^2 x + \cos^2 x)^2 - 3\sin^2 x\cos^2 x) + k((\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x)$
$= 1 - 3\sin^2 x\cos^2 x + k(1 - 2\sin^2 x\cos^2 x)$
$= (1 + k) - (3 + 2k)\sin^2 x\cos^2 x$.
Therefore, if $3 + 2k = 0$ or $k = \frac{-3}{2}$, then $f(x) = 1 + k = \frac{-1}{2}$ for all x and so is constant.
(If $k \neq \frac{-3}{2}$, then we get
$f(0) = 1 + k$
$f(\frac{1}{4}\pi) = (1 + k) - \frac{1}{2}(3 + 2k) + \frac{1}{4}(3 + 2k) = \frac{1}{4} + \frac{1}{2}k$
$f(\frac{1}{6}\pi) = (1 + k) - \frac{1}{4}(3 + 2k) + \frac{1}{16}(3 + 2k) = \frac{7}{16} + \frac{5}{8}k$
which cannot be all equal for any single constant value, so $f(x)$ is not constant if $k \neq \frac{-3}{2}$.)

11. The function $f(x)$ satisfies the equation $f(x) = f(x-1) + f(x+1)$ for all values of x. If $f(1) = 1$ and $f(2) = 3$, what is the value of $f(2014)$?

(a) 3 **(b)** 1 **(c)** -1 **(d)** -3 **(e)** none of the above

The answer is: **(c)**
Solution: Since $f(x) = f(x-1)+f(x+1)$, then $f(x+1) = f(x)-f(x-1)$, and so $f(1) = 1$
$f(2) = 3$
$f(3) = f(2) - f(1) = 3 - 1 = 2$
$f(4) = f(3) - f(2) = 2 - 3 = -1$
$f(5) = f(4) - f(3) = -1 - 2 = -3$
$f(6) = f(5) - f(4) = -3 - (-1) = -2$
$f(7) = f(6) - f(5) = -2 - (-3) = 1 = f(1)$
$f(8) = f(7) - f(6) = 1 - (-2) = 3 = f(2)$
Since the value of f at an integer depends only on the values of f at the two previous integers, then the fact that the first several values form a cycle with $f(7) = f(1)$ and $f(8) = f(2)$ tells us that the values of f will always repeat in sets of 6. Since 2014 is 4 more than a multiple of 6 (as $2014 = 4 + 2010 = 4 + 6(335)$), then $f(2014) = f(2014 - 6(335)) = f(4) = -1$.

12. In the diagram, the circle and the square have the same center O and equal areas. The circle has radius 1 and intersects one side of the square at P and Q. What is the length of PQ?

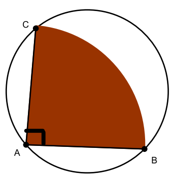


(a) $\sqrt{4-\pi}$ **(b)** 1 **(c)** $\sqrt{2}$ **(d)** $2-\sqrt{\pi}$ **(e)** $4-\sqrt{\pi}$

The answer is: **(a)**

Solution: Since the circle has radius 1, then its area is $\pi(1^2) = \pi$. Since the square and the circle have the same area, then the side length of the square is $\sqrt{\pi}$. Let M be the midpoint of line segment PQ. Since PQ is a chord of the circle, then OM is perpendicular to line segment PQ. Since OM is perpendicular to PQ and O is the center of the square, then OM is half of the length of one of the sides of the square, so $OM = \frac{1}{2}\sqrt{\pi}$.
By the Pythagorean Theorem in $\triangle\ OPM$, we have
$PM^2 = OP^2 - OM^2 = 1^2 - \left(\frac{1}{2}\sqrt{\pi}\right)^2 = 1 - \frac{1}{4}\pi$. Therefore, $PQ = 2PM = 2\sqrt{1-\frac{1}{4}\pi} = \sqrt{4(1-\frac{1}{4}\pi)} = \sqrt{4-\pi}$.

13. In the figure, ABC is a quarter of a circular pizza with center A and radius 20 cm. The piece of pizza is placed on a circular pan with A, B and C touching the circumference of the pan, as shown. What fraction of the pan is covered by the piece of pizza?



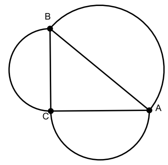
(a) 1/4 **(b)** $1/\sqrt{2}$ **(c)** 1/2 **(d)** $2\sqrt{2}$ **(e)** none of the above

The answer is: **(c)**

Solution: Since ABC is a quarter of a circular pizza with center A and radius 20 cm, then $AC = AB = 20cm$. We are also told that $\angle CAB = 90°$ (one-quarter of 360°). Since $\angle CAB = 90°$ and A, B and C are all on the circumference of the circle, then CB is a diameter of the pan. (This is a property of circles: if X, Y and Z are three points on a circle with $\angle ZXY = 90°$, then YZ must be a diameter of the circle.) Since $\triangle\ CAB$ is right-angled and isosceles, then $CB = \sqrt{2}AC = 20\sqrt{2}$ cm. Therefore, the radius of the circular plate is $1/2CB$ or $10\sqrt{2}$ cm. Thus, the area of the circular pan is $\pi(10\sqrt{2}\text{ cm})^2 = 200\pi\text{ cm}^2$. The area of the slice of pizza is one-quarter of the area of a circle with radius 20 cm, or $1/4\pi(20\text{ cm})^2 = 100\pi\text{ cm}^2$.
Finally, the fraction of the pan that is covered is the area of the slice of pizza divided by the area of the pan, or $\frac{100\pi}{200\pi} = \frac{1}{2}$.

14. A right triangle ABC is given. Semicircles are constructed with the sides of the triangle as diameters, as shown below. Suppose the area of the largest semicircle is 36 and the area of the smallest one is 16. What is the area of the other one?

(a) 20 **(b)** 24 **(c)** 25 **(d)** 26 **(e)** none of the above



The answer is: **(a)**

Solution: Let a, b and c denote the lengths of the three sides, $a = BC$, $b = AC$, and $c = AB$. Now $a^2 + b^2 = c^2$ since the triangle is right. The areas of the three semicircles are $(1/2)\pi(a/2)^2$, $(1/2)\pi(b/2)^2$, $(1/2)\pi(c/2)^2$. Therefore, we have $(1/2)\pi(a/2)^2 + (1/2)\pi(b/2)^2 = \pi(a^2 + b^2)/8 = (\pi/8)c^2$, so the sum of the areas of the two smaller semicircles is the area of the largest one. Thus the area of the middle one is $36 - 16 = 20$.

15. How many positive integers less than 1000 have only odd digits?

(a) 125 **(b)** 155 **(c)** 165 **(d)** 150 **(e)** none of the above

The answer is: **(b)**

Solution: There are five odd digits: 1, 3, 5, 7, 9.
We consider the positive integers less than 1000 in three sets: those with one digit, those with two digits, and those with three digits. There are 5 positive one-digit integers with one odd digit (namely 1, 3, 5, 7, 9). Consider the two-digit positive integers with only odd digits. Such an integer has the form XY where X and Y are digits. There are five possibilities for each of X and Y (since each must be odd). Therefore, there are $5 \times 5 = 25$ two-digit positive integers with only odd digits. Consider the three-digit positive integers with only odd digits. Such an integer has the form XYZ where X, Y and Z are digits. There are five possibilities for each of X, Y and Z (since each must be odd). Therefore, there are $5 \times 5 \times 5 = 125$ three-digit positive integers with only odd digits. In total, there are $5 + 25 + 125 = 155$ positive integers less than 1000 with only odd digits.

16. The tips of a five-pointed star are to be painted red, white and blue. How many ways can this be done if no adjacent points can be the same color?
(a) 30 **(b)** 10 **(c)** 40 **(d)** 45 **(e)** none of the above

The answer is: **(a)**
Solution: Let the 5 points be $ABCDE$. Note that to paint the 5 points in the prescribed fashion, 2 colors must be used twice and one color just once. Also, note that if point A is the color that is used only once, then there are only two ways to paint the star. Consider one such set of colors, where red is the single color used. Then if A is painted red there are only two ways to paint the rest of the star. A similar situation occurs if the points $B, C, D, or E$ are painted with the red. For this set of colors there are 10 ways to paint the star. Since there are three different sets of colors, there are a total of 30 ways to paint the star.

17. If $x > 0$, $x \neq 1$, $a = \log_2(x)$ and $b = \log_x(7)$, what is $\log_{14}(x)$?
(a) ab **(b)** $a + b$ **(c)** $b - a$ **(d)** a/b **(e)** none of the above

The answer is: **(e)**
Solution 1: Note that $b = \log_x 7 = \frac{\log_2 7}{\log_2 x} = \frac{\log_2 7}{a}$, i.e., $ab = \log_2 7$. Hence

$$\log_{14} x = \frac{\log_2 x}{\log_2 14} = \frac{a}{\log_2 2 + \log_2 7} = \frac{a}{1 + ab}.$$

Solution 2: Alternatively, one can convert logarithms to exponentials, we have $a = \log_2(x)$ implies $2^a = x$. So $2 = x^{(1/a)}$. Also $b = \log_x(7)$ implies $7 = x^b$. Thus $14 = x^{(1/a)} \cdot x^b = x^{(1/a+b)} = x^{(1+ab/a)}$. So $14^{(a/1+ab)} = x$ and hence $\log_{14}(x) = a/1 + ab$.

18. Suppose that a, b, c, and d are positive integers that satisfy the equations

$$ac + bd = 34$$

$$ad + bc = 43$$

What is the value of $a + b + c + d$?

(a) 19 **(b)** 18 **(c)** 17 **(d)** 16 **(e)** none of the above

The answer is: **(b)**

Solution: Adding the two equations, we obtain

$$ac + bd + ad + bc = 77$$

$$ac + ad + bc + bd = 77$$

$$a(c + d) + b(c + d) = 77$$

$$(a + b)(c + d) = 77$$

Since each of a, b, c and d is a positive integer, then $a + b$ and $c + d$ are each positive integers and are each at least 2. Since the product of $a + b$ and $c + d$ is $77 = 7 \times 11$ (with 7 and 11 both prime), then one must equal 7 and the other must equal 11. Therefore, $a + b + c + d = 7 + 11 = 18$. (We can check with some work that $(a, b, c, d) = (5, 2, 4, 7)$ is a solution to the system.)

19. What is the coefficient of x^3 in the polynomial $(1 + x + x^2)^{20}$?

(a) 20 **(b)** 380 **(c)** 1520 **(d)** 2340 **(e)** none of the above

The answer is: **(c)**

Solution: Let

$(1+x+x^2)^{20} = [1+(x+x^2)]^{20} = 1^{20}+20\cdot 1^{19}(x+x^2)+190\cdot 1^{18}(x+x^2)^2+1140\cdot 1^{17}(x+x^2)^3+\ldots$

The terms involving x^3 are: $(190 \cdot 2)x^3 + 1140x^3 = 1520x^3$.

20. Suppose $f(x) = ax + b$ and a and b are real numbers. We define $f_1(x) = f(x)$ and $f_{n+1}(x) = f(f_n(x))$ for all positive integers n. If $f_7(x) = 128x + 381$, what is the value of $a + b$?

(a) 7 **(b)** 5 **(c)** 3 **(d)** 2 **(e)** 1

The answer is: **(b)**

Solution: From the definition, $f_n(x) = a^n x + (a^{n-1} + a^{n-2} + \dots a + 1)b = a^n x + \frac{a^n - 1}{a-1} \cdot b$. From $f_7(x) = 128x + 381$, we get $a^7 = 128$, and $\frac{a^7-1}{a-1} \cdot b = 381$, therefore, $a = 2, b = 3$, and hence, $a + b = 5$.

21. Let a, b and c be real numbers which satisfy the three equations below.

$$a + \frac{1}{bc} = \frac{1}{5}, \qquad b + \frac{1}{ac} = \frac{-1}{15}, \qquad c + \frac{1}{ab} = \frac{1}{3}.$$

What is the value of the quotient $\frac{c-b}{c-a}$?

(a) -5 **(b)** -3 **(c)** 1 **(d)** 3 **(e)** 5

The answer is: **(d)**

Solution 1: Proceed with direct resolution of the three equations to get $bc/5 = -ac/15 = ab/3$. It then follows that $a = -3b$ and $c = -5b$. Now substitute a and c in $(c - b)/(c - a)$ to get 3.

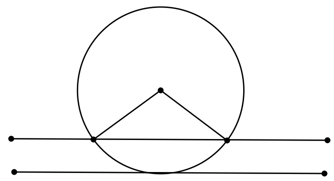
Solution 2: Alternatively, let $S = \frac{abc+1}{abc}$. The equations can be rewritten as $aS = \frac{1}{5}$, $bS = \frac{-1}{15}$, and $cS = \frac{1}{3}$. Since $S \neq 0$, we have $\frac{c-b}{c-a} = \frac{cS-bS}{cS-aS} = 3$. (Note that real numbers a, b, c satisfying the above equation do exist. They are $a = 3u, b = -u, c = 5u$ where u is the unique real solution of the cubic equation $15x^3 - x^2 - 1 = 0$.)

22. Given are two parallel lines of distance 1 apart and a circle of radius 2. The circle is tangent to one of the lines and cuts the other line. The area of the circular cap between the two parallel lines is $a\frac{\pi}{3} - b\sqrt{3}$. Find the sum $a + b$ of the two integers a and b.

(a) 3 **(b)** 4 **(c)** 5 **(d)** 6 **(e)** 7

The answer is: **(c)**

Solution: The cap is a segment minus a triangle. The central angle of the segment is 120°. The circular segment covers one third of the circle and hence has area $\frac{4\pi}{3}$. Hence $a = 4$. The triangle is isosceles with height 1 and two congruent sides of length 2. Its third side has length $2\sqrt{3}$ and hence its area is $\sqrt{3}$. Hence $b = 1$ and $a + b = 5$.



23. The average of three consecutive multiples of 3 is a.
The average of four consecutive multiples of 4 is $a + 27$.
The average of the smallest and largest of these seven integers is 42.
The value of a is

(a) 15 **(b)** 18 **(c)** 24 **(d)** 27 **(e)** none of the above

The answer is: **(d)**

Solution: Since the average of three consecutive multiples of 3 is a, then a is the middle of these three integers, so the integers are $a-3, a, a+3$. Since the average of four consecutive multiples of 4 is $a + 27$, then $a + 27$ is halfway in between the second and third of these multiples (which differ by 4), so the second and third of the multiples are $(a+27)-2 = a+25$ and $(a + 27) + 2 = a + 29$, so the four integers are $a + 21, a + 25, a + 29, a + 33$.
(We have used in these two statements the fact that if a list contains an odd number of integers, then there is a middle integer in the list, and if the list contains an even number of integers, then the "middle" integer is between two integers from the list.)
The smallest of these seven integers is $a - 3$ and the largest is $a + 33$. The average of these two integers is $\frac{1}{2}(a - 3 + a + 33) = \frac{1}{2}(2a + 30) = a + 15$. Since $a + 15 = 42$, it follows that $a = 27$.

24. Let AB and CD be two chords of a circle that intersect at a point P. Suppose that $AP = 4, PB = 6, CP = 2, PD = 12$ and $\angle APC = 90°$. What is the radius of the circle?

(a) $4\sqrt{3}$ **(b)** $3\sqrt{6}$ **(c)** 8 **(d)** $5\sqrt{2}$ **(e)** none of the above

The answer is: **(d)**

Solution: Let M be the midpoint of the chord AB, N the midpoint of the chord CD, and let O be the center of the circle. Then $AM = BM = 5$, and $CN = ND = 7$, so $NP = 5$. Moreover, OM is perpendicular to AB, and ON is perpendicular to CD. Thus, $PMON$ is a rectangle, and $NP = OM = 5$. Applying the Pythagorean Theorem to $\triangle\ OBM$ we get $OB = 5\sqrt{2}$.

25. For how many integers n is the value of $\sqrt{\frac{n}{50-n}}$ is an integer?

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** 5

The answer is: **(e)**

Solution: The expression $\sqrt{\frac{n}{50-n}}$ is an integer whenever $\frac{n}{50-n}$ is a perfect square. Let $\frac{n}{50-n} = k^2$. Then $n = \frac{50k^2}{k^2+1}$. Since $(k^2+1) - k^2 = 1$, it follows that $k^2 + 1$ and k^2 are relatively prime unless $k = 0$. Now because n is an integer and $n = \frac{50k^2}{k^2+1}$, we see that $k^2 + 1$ must divide $50 = 5 \times 5 \times 2$. This happens when $k = 0, 1, 2, 3$ or 7.

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Dear Teachers/Students:

If you do have any suggestions about the competition, or if you have different solutions to any of this year's problems, please send them by mail or e-mail to

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Remember to visit us for information about past competitions at
http://www.svsu.edu/matholympics/

The SVSU Math Olympic Committee would like to express his gratitude to all participants.

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