## SAGINAW VALLEY STATE UNIVERSITY
## SOLUTIONS OF 2014 MATH OLYMPICS LEVEL I

**1.** If $2^x = 2(16^{12}) + 2(8^{16})$, what is the value of $x$?
**(a)** 50 **(b)** 37 **(c)** 28 **(d)** 25 **(e)** none of the above

The answer is: **(a)**
**Solution:** Simplifying using exponent rules,
$2(16^{12}) + 2(8^{16}) = 2((2^4)^{12}) + 2((2^3)^{16}) = 2(2^{48}) + 2(2^{48}) = 4(2^{48}) = 2^2(2^{48}) = 2^{50}$.
Therefore, since $2^x = 2(16^{12}) + 2(8^{16}) = 2^{50}$, then $x = 50$.

**2.** Suppose the operation $*$ is defined on the set of integers by $a * b = 1 - a - b$. Then for every two integers $a$ and $b$, the value of $a * (a * b)$ is the same as
**(a)** 0 **(b)** 2 **(c)** $a$ **(d)** $b$ **(e)** none of the above

The answer is: **(d)**
**Solution:** Using the definition of $*$, we have
$a * (a * b) = a * (1 - a - b) = 1 - a - (1 - a - b) = 1 - a - 1 + a + b = b$.

**3.** Determine all values of $k$, with $k \neq 0$, for which the parabola $y = kx^2 + (5k + 3)x + (6k + 5)$ has its vertex on the $x$-axis.
**(a)** -3, -3 **(b)** -2, -6 **(c)** -1, -9 **(d)** 0, -12 **(e)** 1, 3

The answer is: **(c)**
**Solution:** For the parabola to have its vertex on the $x$-axis, the equation
$y = kx^2 + (5k + 3)x + (6k + 5) = 0$ must have two equal real roots. That is, its discriminant must equal 0, and so $(5k + 3)^2 - 4k(6k + 5) = 0$. Thus,
$25k^2 + 30k + 9 - 24k^2 - 20k = 0$, i.e., $k^2 + 10k + 9 = 0$. So $(k + 1)(k + 9) = 0$ and therefore, $k = -1$ or $k = -9$.

**4.** Nana and Boris took the Math Olympics Contest. If we double Nana's score we get 60 more than Boris's score. If we double Boris's score we get 90 more than Nana's score. Determine the average of their two scores.
**(a)** 90 **(b)** 80 **(c)** 70 **(d)** 60 **(e)** none of the above

The answer is: **(e)**

**Solution:** Let $N$ be Nana's score and $B$ be Boris's score. Since two times Nana's score was 60 more than Boris's score, then $2N = B + 60$. Since two times Boris's score was 90 more than Nana's score, then $2B = N+90$. Adding these two equations, we obtain $2N + 2B = B + N + 150$ or $N + B = 150$ or $\frac{N+B}{2} = 75$. Therefore, the average of their two scores was 75.
(Note that we didn't have to solve for their individual scores. Alternatively, you may solve each individual score to get $N = 70$ and $B = 80$ and the average of 70 and 80 is 75)

**5.** The units digit in the product $(5^2 + 1)(5^3 + 1)(5^{23} + 1)$ is

**(a)** 3 **(b)** 5 **(c)** 6 **(d)** 2 **(e)** 1

The answer is: **(c)**

**Solution:** Since $5^2$, $5^3$ and $5^{23}$ all end in 5, then $5^2 + 1$, $5^3 + 1$ and $5^{23} + 1$ all end in 6. When we multiply these three numbers together their product must also end in a 6.

**6.** The number of integers $x$ for which the value of $\frac{-6}{x+1}$ is an integer is

**(a)** 2 **(b)** 4 **(c)** 6 **(d)** 8 **(e)** 9

The answer is: **(d)**

**Solution:** Since $x$ is an integer, then $x+1$ is an integer. Since $\frac{-6}{x+1}$ is to be integer, then $x+1$ must be a divisor of -6. Thus, there are 8 possible values for $x+1$, namely -6, -3, -2, -1, 1, 2, 3, and 6. This gives 8 possible values for $x$, namely -7, -4, -3, -2, 0, 1, 2, and 5.

**7.** On Monday, 10% of the students at SVSU were absent and 90% were present. On Tuesday, 10% of those who were absent on Monday were present and the rest of those absent on Monday were still absent. Also, 10% of those who were present on Monday were absent and the rest of those present on Monday were still present. What percentage of the students at SVSU were present on Tuesday?

**(a)** 99% **(b)** 91% **(c)** 90% **(d)** 88% **(e)** 82%

The answer is: **(e)**

**Solution:** Suppose that there are 10000 students at SVSU. On Monday, there were thus 1000 students absent and 9000 students present. On Tuesday, 10% of

the 9000 students who were present on Monday, or 0.1(9000) = 900 students, were absent. The remaining 9000 - 900 = 8100 students who were present on Monday were still present on Tuesday. Similarly, 10% of the 1000 students who were absent on Monday, or 0.1(1000) = 100 students, were present on Tuesday. The remaining 1000 - 100 = 900 students who were absent on Monday were still absent on Tuesday. Thus, there were 8100 + 100 = 8200 students present on Tuesday, or $\frac{8200}{10000} \times 100\%$ = 82% of the whole student population.
(Note that this reasoning is independent of the initial choice of 10000 students, you may assume that the initial population is $x$. Note also that 10000 is the actual approximative number of students attending SVSU this year. You are welcome to join us in the coming years!)

**8.** Let $n$ be the smallest positive integer whose digits have a product of 2000. The sum of the digits of $n$ is

**(a)** 21 **(b)** 23 **(c)** 25 **(d)** 27 **(e)** 29

The answer is: **(c)**

**Solution:** Since $2000 = 2^4 \times 5^3$, the smallest possible positive integer satisfying the required conditions is 25,558 which gives the sum 2 + 5 + 5 + 5 + 8 = 25.
(Note that another answer might be 23 since 44,555 satisfies the given conditions. However, since $25,558 < 44,555$ and the question requires the smallest number then the answer must be 25 and not 23.)

**9.** Find the sum of all values of $x$ that satisfy $|x+1| + 3|x-2| + 5|x-4| = 20$.

**(a)** 2 **(b)** 5 **(c)** 6 **(d)** 9 **(e)** 11

The answer is: **(c)**

**Solution:** Consider the four cases, $x < -1, -1 < x < 2, 2 < x < 4$, and $4 < x$. Each of these gives rise to a linear equation in $x$. Just two of these have solutions in the appropriate intervals, $x = 1$ and $x = 5$. Their sum is 6.

**10.** You have 10 coins, all of different weights and you can weigh them only in pairs in a two-pan balance. What is the minimal numbers of weighings needed to find the heaviest coin?

**(a)** 45 **(b)** 12 **(c)** 10 **(d)** 9 **(e)** 5

The answer is: **(d)**

**Solution:** Each time you compare the new coin with the heaviest form the previous pair. With each pair weighed, the best you can do is eliminate one coin at a time. So 9 is the minimal numbers of pairs weighed.

**11.** In his last will, a farmer asked that his horses be distributed among his four sons. The oldest was to get one third of the herd, the second oldest, one fourth of the herd, and each of the two youngest ones was to get one fifth of the herd. When the sons read the will, they were puzzled because none of them were going to get an integer number of horses. At that moment, they discovered that a baby horse had just been born. Each son would receive an integer number of horses, but the baby horse would be left over. How many horses did the farmer have originally?

**(a)** 29 **(b)** 59 **(c)** 89 **(d)**119 **(e)** 239

The answer is: **(b)**

**Solution:** Suppose the farmer originally had $N$ horses. Splitting $N+1$ horses as described in the will and recalling that one horse is not distributed leads to

$$\frac{N+1}{3}+\frac{N+1}{4}+\frac{2(N+1)}{5}=N$$

Adding the fractions yields

$$\frac{59(N+1)}{60}=N$$

Hence $N=59$.

**12.** An athlete covers three consecutive miles by swimming the first, running the second and cycling the third. He runs twice as fast as he swims and cycles one and a half times as fast as he runs. He takes ten minutes longer than he would if he had cycled the whole three miles. How many minutes does he take?

**(a)** 16 minutes **(b)** 22 minutes **(c)** 30 minutes
**(d)** 46 minutes **(e)** none of the above

The answer is: **(b)**

**Solution 1:** Let $T$ denote the time in hours required to swim a mile. Then the time required to run a mile is $\frac{T}{2}$, and the time needed to cycle a mile is $\frac{T}{3}$. It follows that $T+\frac{1}{2}T+\frac{1}{3}T-\frac{1}{6}=3(\frac{1}{3}T)$, so $T=\frac{1}{5}$, and $T+\frac{1}{2}T+\frac{1}{3}T=\frac{11}{6}\cdot\frac{1}{5}=\frac{11}{30}$ hours, which is 22 minutes.

**Solution 2:** Alternatively, let $s$ be the swimming speed, $r$ the running speed and $c$ the cycling speed in miles per minute. Then his total time for the three miles is $t=(1/s)+(1/r)+(1/c)=10+(3/c)$. Also, $r=2s$ and $c=1.5r$. So $c=3s$,

$1/s = 3/c$ and $1/r = 3/2c$. Subbing in yields $5/2c = 10$ and $1/c = 4$. So the total time is $10 + 3(4) = 22$ minutes.

**13.** Given that $a$ and $b$ are positive real numbers with $a + b = 4$, what is the minimum value of $\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)$?

**(a)** 2 **(b)** 3 **(c)** 4 **(d)** $\frac{8}{3}$ **(e)** $\frac{9}{4}$

The answer is: **(e)**

**Solution:** We have
$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right) = \frac{1+a+b+ab}{ab} = 1 + \frac{5}{ab}$.
Thus, we need to maximize $ab = a(4 - a) = 4a - a^2$. The maximum of a parabola that opens downwards is at its vertex. So the maximum of $4a - a^2$ is at $a = 2$ and hence $b = 2$.

**14.** You own eleven pairs of socks, all different, and all of the socks are individually jumbled in a drawer. One morning you rummage through the drawer and continue to pull out socks until you have a matching pair. How many socks must you pull out to guarantee having a matching pair?

**(a)** 3 **(b)** 6 **(c)** 11 **(d)** 12 **(e)** none of the above

The answer is: **(d)**

**Solution:** You might be unlucky and have the first eleven socks all different, but then the 12th has to match one of them.

**15.** Given that $f(x) = (x^5 - 1)(x^3 + 1), g(x) = (x^2 - 1)(x^2 - x + 1)$, and $h(x)$ is a polynomial such that $f(x) = g(x)h(x)$, what is the value of $h(1)$?

**(a)** 0 **(b)** 2 **(c)** 3 **(d)** 5 **(e)** undefined

The answer is: **(d)**

**Solution:** We have, after factorization,

$$f(x) = (x^5 - 1)(x^3 + 1) = (x - 1)(x^4 + x^3 + x^2 + x + 1)(x + 1)(x^2 - x + 1).$$

Since $(x - 1)(x + 1) = x^2 - 1$, it follows that $f(x) = (x^2 - 1)(x^4 + x^3 + x^2 + x + 1)(x^2 - x + 1) = (x^4 + x^3 + x^2 + x + 1)g(x)$. Hence $h(x) = x^4 + x^3 + x^2 + x + 1$ and $h(1) = 5$.

**16.** The set of all $x$ such that $(|x| - 2)(1 + x) > 0$ is exactly

**(a)** $|x| > 2$ **(b)** $-2 < x < 1$ **(c)** $x > 2$ **(d)** $-2 < x < -1$
**(e)** none of the above

The answer is: **(e)**

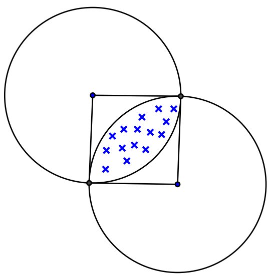
**Solution:** The inequality $(|x|-2)(1+x) > 0$ is satisfied if both factors are positive or if both are negative. Both are positive if $x > 2$ and both are negative if $-2 < x < -1$. So all $x$ such that $x > 2$ or $-2 < x < -1$ satisfies the given inequality.

**17.** A doll that is 4 inches tall represents a 64 inch person. A doll house is to be made using the same scaling factor that is an exact model of a real house. If a rectangular rug in the real house has an area of $32ft^2$, what size will the rug be in the doll house?
**(a)** $18in^2$ **(b)** $24in^2$ **(c)** $512in^2$ **(d)** $42.66in^2$ **(e)** $2in^2$

The answer is: **(a)**

**Solution:** The linear scaling factor is found by:
doll length = person length × scaling factor. So
4 in = 64 in × scaling factor. Thus, scaling factor = 4in/64in = 1/16.
Since an area is a length times width, it will be multiplied by the square of the scaling factor. The size of the doll house rug will be
$(32ft^2)\frac{12^2in^2}{ft^2}\left(\frac{1}{16}\right)^2 = \frac{32\cdot 12^2}{16^2}in^2 = 18in^2$.

**18.** What is the area of the region common to two unit circles with centers that are $\sqrt{2}$ apart?



**(a)** $\frac{\pi}{2} - 1$ **(b)** $\frac{1}{2}(1 - \frac{\pi}{4})$ **(c)** $1 - \frac{\pi}{4}$ **(d)** $\frac{\pi}{2}$ **(e)** none of the above

The answer is: **(a)**

**Solution:** Since the area of the square is 1 and the area of the sector is $\pi/4$, half the area of the unshaded region inside the square is $1 - \frac{\pi}{4}$. So the shaded area is

$$1 - 2(1 - \frac{\pi}{4}) = \frac{\pi}{2} - 1.$$

**19.** Suppose that $f(x) = ax + b$ where $a$ and $b$ are real numbers. Given that $f(f(f(x))) = 8x + 21$, what is the value of $a + b$?

**(a)** 6 **(b)** 5 **(c)** 4 **(d)** 3 **(e)** none of the above

The answer is: **(b)**

**Solution:** Since $f(x) = ax + b$, we have $f(f(x)) = a(ax + b) + b = a^2x + b(a + 1)$ and $f(f(f(x))) = a\left(a^2x + b(a + 1)\right) + b = a^3x + b(a^2 + a + 1)$. Since this must equal $8x + 21$, we deduce that $a = 2$ and $b = 21/7 = 3$. Hence, $a + b = 5$.

**20.** Curt's car gets 3 more miles per gallon during highway driving than it does during city driving. On a recent trip, Curt drove 136 miles on the highway and 155 miles in the city, using a total of 9 gallons of gasoline. How many miles per gallon does Curt's car get during city driving?

**(a)** 30 **(b)** 31 **(c)** 32 **(d)** 34 **(e)** 35

The answer is: **(b)**

**Solution:** Let $x$ denote the number of miles per gallon Curt gets during city driving so that $x + 3$ is the number of miles per gallon Curt gets during highway driving. Then Curt used $\frac{155}{x}$ gallons in the city and $\frac{136}{x+3}$ gallons on the highway. Thus,

$$\frac{155}{x} + \frac{136}{x + 3} = 9 \Rightarrow 9x^2 - 264x - (3 \times 155) = 0.$$

Factoring a 3 from the last equation gives $3x^2 - 88x - 155 = 0$. Hence, $(3x + 5)(x - 31) = 0$ so that $x = 31$.

**21.** The area bounded by two concentric circles is $5\pi$ square centimeters. The difference between the radii of the circles is 1 centimeter. What is the radius of the smaller circle, in centimeters?

**(a)** 6 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** 4

The answer is: **(c)**

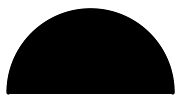
**Solution:** If $r$ is the radius of the smaller circle, the area of the annular region is $(r + 1)^2\pi - r^2\pi = (2r + 1)\pi$. The solution of the equation $2r + 1 = 5$ is $r = 2$.

Alternatively, let $s$ be the radius of the larger circle Then $5\pi = s^2\pi - r^2\pi = (s - r)(s + r)\pi$. Since $s - r = 1, s + r = 5$, it follows that $s = 3$ and $r = 2$.

**22.** Point $A$ on a coordinate plane has coordinates $(2, 0)$. Point $B$ lies in the first quadrant on the line $x - y = 1$. If the distance from point $A$ to point $B$ is 5 units, what is the sum of the coordinates of point $B$?
**(a)** $\sqrt{41}$ **(b)** 9 **(c)** 3 **(d)** 11 **(e)** none of the above

The answer is: **(b)**
**Solution:** On the line $x - y = 1$, i.e., $y = x - 1$. Plugging into distance formula (and squaring both sides) you get: $(x - 2)^2 + (x - 1)^2 = 25$. Simplifying gives: $x^2 - 3x - 10 = 0$. Which can be solved by factoring to get $x = 5$ or $-2$. Since it is in the first quadrant $x = 5$, so $y = 5 - 1 = 4$.

**23.** In the diagram, the perimeter of the semicircular region is 20. (The perimeter includes both the semicircular arc and the diameter.) The area of the region is



**(a)** $\frac{1}{2}\pi\left(\frac{20}{\pi+1}\right)^2$ **(b)** $\frac{1}{2}\pi\left(\frac{10}{\pi+1}\right)^2$ **(c)** $200(\pi + 4)$ **(d)** $400(\pi + 4)$
**(e)** none of the above

The answer is: **(e)**
**Solution:** Suppose that the radius of the region is $r$. The length of the semi-circle is half of the circumference, or $\frac{1}{2}(2\pi r) = \pi r$. Thus, the perimeter of the shaded region is $\pi r + 2r$. Since the perimeter is 20, then $\pi r + 2r = 20$ or $r(\pi + 2) = 20$ or $r = \frac{20}{\pi+2}$. Thus, the area of the semi-circle is half of the area of a circle with this radius, or $\frac{1}{2}\pi\left(\frac{20}{\pi+2}\right)^2$.

**24.** The area of a circle circumscribed about a regular hexagon is $200\pi$. What is the area of the hexagon?

**(a)** $300\sqrt{3}$ **(b)** $600\sqrt{3}$ **(c)** $60\sqrt{3}$ **(d)** 600 **(e)** 1200

The answer is: **(a)**

**Solution:** A circle with an area of $200\pi = \pi r^2$ has a radius of $10\sqrt{2}$. A regular hexagon is made up of six equilateral triangles each with side equal to the radius of the circle. Since a hexagon has six sides the sum of the angles is (4 - 2)180 = 720, so that each interior angle has measure 120°. The side of the triangle bisects this angle resulting in a 30-60-90 triangle. Using this information, the area of each triangle is computed to be $50\sqrt{3}$ so the area of the hexagon is $6 \times 50\sqrt{3} = 300\sqrt{3}$.

**25.** For how many integers $n$ is the value of $\frac{n}{50-n}$ the square of an integer?
**(a)** 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** 5

The answer is: **(e)**

**Solution:** Let $\frac{n}{50-n} = k^2$. Then $n = \frac{50k^2}{k^2+1}$. Since $(k^2+1) - k^2 = 1$, it follows that $k^2 + 1$ and $k^2$ are relatively prime unless $k = 0$. Now because $n$ is an integer and $n = \frac{50k^2}{k^2+1}$, we see that $k^2 + 1$ must divide $50 = 5 \times 5 \times 2$. This happens when $k = 0, 1, 2, 3$ or 7.

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Dear Teachers/Students:

If you do have any suggestions about the competition, or if you have different solutions to any of this year's problems, please send them by mail or e-mail to

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Remember to visit us for information about past competitions at
*http://www.svsu.edu/matholympics/*

The SVSU Math Olympic Committee would like to express his gratitude to all participants.

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