SAGINAW VALLEY STATE UNIVERSITY
SOLUTIONS OF 2013 MATH OLYMPICS LEVEL II

1. The following inequalities hold for all positive integers n:

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{4n+1}} < \sqrt{n} - \sqrt{n-1}.$$

What is the greatest integer which is less than

$$\sum_{n=1}^{24} \frac{1}{\sqrt{4n+1}}?$$

(a) 2 **(b)** 3 **(c)** 4 **(d)** 5 **(e)** 6

The answer is: **(c)**

Solution: The inequalities

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{4n+1}} < \sqrt{n} - \sqrt{n-1}$$

imply

$$\begin{aligned}
\sum_{n=1}^{24} \frac{1}{\sqrt{4n+1}} &< \sum_{n=1}^{24} \sqrt{n} - \sqrt{n-1} \\
&= (\sqrt{1} - \sqrt{0}) + (\sqrt{2} - \sqrt{1}) + \ldots + (\sqrt{24} - \sqrt{23}) \\
&= \sqrt{24} < 5
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=1}^{24} \frac{1}{\sqrt{4n+1}} &> \sum_{n=1}^{24} \sqrt{n+1} - \sqrt{n} \\
&= (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \ldots + (\sqrt{25} - \sqrt{24}) \\
&= \sqrt{25} - \sqrt{1} = 4.
\end{aligned}$$

Hence, the answer is 4.

2. Find the closest integer to $a+b$ if

$$a = \frac{1}{4} + \frac{3}{8} + \frac{5}{12} + \ldots + \frac{1005}{2012}$$

and

$$b = \frac{5}{8} + \frac{7}{12} + \frac{9}{16} + \ldots + \frac{1009}{2016}.$$

(a) 503 **(b)** 502 **(c)** 1 **(d)** 0 **(e)** none of the above

The answer is: **(a)**

Solution: Adding the two numbers and grouping the terms as shown we compute

$$a+b = \frac{1}{4} + \left(\frac{3}{8} + \frac{5}{8}\right) + \left(\frac{5}{12} + \frac{7}{12}\right) + \ldots \left(\frac{1005}{2012} + \frac{1007}{2012}\right) + \frac{1009}{2016}.$$

We use the fact that $\left(\frac{3}{8} + \frac{5}{8}\right) = \left(\frac{5}{12} + \frac{7}{12}\right) = \ldots = \left(\frac{1005}{2012} + \frac{1007}{2012}\right) = 1$ and the fact that there are $(2012/4) - 1$ such terms, to get $a + b = 502 + \frac{1}{4} + \frac{1009}{2016}$. Now, since $\frac{1009}{2016} > \frac{1}{4}$, it follows that $\frac{1}{4} + \frac{1009}{2016} > \frac{1}{2}$. Thus the closest integer to $a+b$ is 503.

3. How many 4 digit numbers with first digit 2 have exactly one pair of two identical digits (like 2011, or 2012)?

(a) 216 **(b)** 108 **(c)** 432 **(d)** 54 **(e)** none of the above

The answer is: **(c)**

Solution: The repeated number is either a 2 or a digit in the set $\{0, 1, 3, 4, 5, 6, 7, 8, 9\}$. Let A be the set of 4 digit numbers starting with 2 which have an additional 2 among its digits and no other repeated digits. We have 3 ways of choosing where to place the 2. The additional two digits can be assigned in 9×8 ways. Therefore, the number of elements of the set A is $|A| = 3 \times 72 = 216$. Let B be the set of 4 digit numbers starting with 2 which have a repeated digit among the remaining three digits. There are three ways of choosing where the two repeated digits among $\{0, 1, 3, 4, 5, 6, 7, 8, 9\}$ go. Since there are 9 possible digits to choose from, we have 27 ways of placing these digits. Once we have chosen this digit there are 8 remaining digits left to place in the other digit. Hence there are $27 \times 8 = 216$ ways of placing the digits in B. Since A and B are disjoint sets we can compute the number of elements which are either in A or in B as follows $|A \cup B| = |A| + |B| = 216 + 216 = 432$.

4. Which of the following is an algebraic expression of $\sin(\sin^{-1} x + \cos^{-1} y)$, for $-1 \le x \le 1$ and $-1 \le y \le 1$?

(a) $xy + \sqrt{1-x^2}\sqrt{1-y^2}$ **(b)** $y\sqrt{1-x^2} + x\sqrt{1-y^2}$ **(c)** $2\sqrt{1-x^2}\sqrt{1-y^2}$
(d) $xy \pm \sqrt{1-x^2}\sqrt{1-y^2}$ **(e)** none of the above

The answer is: **(a)**

Solution: Let $\alpha = \sin^{-1} x$ and $\beta = \cos^{-1} y$, i.e., $x = \sin\alpha$ with $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ and $y = \cos\beta$ with $0 \le \beta \le \pi$ respectively. Since $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, we have $\cos\alpha \ge 0$. As a result,
$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-x^2}$.
Similarly, since $0 \le \beta \le \pi$, we have $\sin\beta \ge 0$. Then
$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-y^2}$.
Hence
$\sin(\sin^{-1} x + \cos^{-1} y) = \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = xy + \sqrt{1-x^2}\sqrt{1-y^2}$.

5. The Bayes family has two cars. One of them has a 90% chance of starting on any given morning, and the other has an 80% chance. On any given morning, what is the probability that at least one of the cars will start?

(a) .72 **(b)** .9 **(c)** .7 **(d)** .98 **(e)** none of the above

The answer is: **(d)**

Solution: The probability that the first car won't start is .1. The probability that the second car won't start is .2. So the probability that they both fail to start is .02. (We assume the events are independent.) At least one starting is the complement of both failing, so the probability that at least one starts is $1 - .02 = .98$.

6. In the sum below, the letter $F = 0$, and the other letters represent the digits $1, 2, 3, 4, 5,$ or 6, with each digit used exactly once. The 2-digit integer AB is a prime number. What is the value of $A + B$?

$$\begin{array}{rr} & AB \\ + & CD \\ \hline & EFG \end{array}$$

(a) 7 **(b)** 5 **(c)** 4 **(d)** 3 **(e)** none of the above

The answer is: **(a)**

Solution: Since AB is prime, B is either 1 or 3. Also, $A + C \leq 11$, so $E = 1$. It follows that $B = 3$. Since $D \leq 6$, $B + D < 10$ and $A + C = 10$. We deduce that A and C must be 4 and 6 in some order. Since 63 is not prime, $AB = 43$. The answer is $4 + 3 = 7$.

7. Which of the following functions is the inverse to the function $f(x) = x^{15} - 6x^{10} + 12x^5 - 8$?

(a) No inverse exists because the function is not one-to-one **(b)** $f^{-1}(x) = \sqrt[15]{\frac{x+8}{x^{15}+6x^{10}+12}}$

(c) $f^{-1}(x) = \sqrt[5]{\sqrt[3]{x} + 2}$ **(d)** $f^{-1}(x) = \sqrt[15]{\frac{\sqrt[10]{x}}{6} + \frac{\sqrt[5]{x}}{12} + 8}$

(e) none of the above

The answer is: **(c)**

Solution: We can factor $f(x)$ to get $f(x) = (x^5 - 2)^3$. To find the inverse we set $y = (x^5 - 2)^3$, then we switch the variables x and y to get $x = (y^5 - 2)^3$; and then solve for y. Take the cube root of both sides, add 2 to both sides and then take the fifth root of both sides to get $y = \sqrt[5]{\sqrt[3]{x} + 2}$.

8. A finite sequence $a_0, a_1, \ldots, a_n$ of integers is called a *curious sequence* if it has the property that for every $k = 0, 1, 2, \ldots, n$, the number of times k appears in the sequence is a_k. For example, $a_0 = 1$, $a_1 = 2, a_2 = 1, a_3 = 0$ forms a curious sequence. Let $a_0, a_1, \ldots, a_{100}$ be a curious sequence. What is the value of the sum $\sum_{k=0}^{100} a_k$?

(a) 201 **(b)** 101 **(c)** 200 **(d)** 100 **(e)** none of the above

The answer is: **(b)**

Solution: The sum $\sum_{k=0}^{100} a_k$ is simply the sum of the number of times one of the numbers $0, 1, 2, \ldots, 100$ occurs in the sequence. On the other hand, each of $a_0, a_1, \ldots, a_{100}$ is one of the numbers $0, 1, 2, \ldots, 100$. Thus, the sum is the number of terms in the sum, namely 101.

9. $\sin 3\theta =$?

(a) $3(\cos^2\theta - \sin^2\theta)$ **(b)** $\sin^2\theta\cos\theta$ **(c)** $3\sin\theta$ **(d)** $3\sin\theta - 4\sin^3\theta$

(e) none of the above

The answer is: **(d)**

Solution: It is easy to see that
$\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos\theta + \cos 2\theta \sin\theta = (2\sin\theta\cos\theta)\cos\theta + (2\cos^2\theta - 1)\sin\theta = 2\cos^2\theta\sin\theta + 2\cos^2\theta\sin\theta - \sin\theta = 4\cos^2\theta\sin\theta - \sin\theta = 4(1-\sin^2\theta)\sin\theta - \sin\theta = 3\sin\theta - 4\sin^3\theta$.

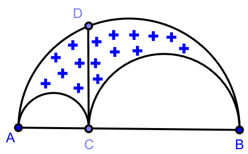
10. If $f\left(\frac{x}{x-1}\right) = \frac{1}{x}$ for all $x \neq 0, 1$ and $0 < \theta < \pi/2$, then $f(\sec^2\theta) =$
(a) $\cot^2\theta$ **(b)** $\tan^2\theta$ **(c)** $\cos^2\theta$ **(d)** $\sin^2\theta$
(e) none of the above

The answer is: **(d)**
Solution: If $\frac{x}{x-1} = \sec^2\theta$, then $x = x\sec^2\theta - \sec^2\theta$. It follows that $x(\sec^2\theta - 1) = \sec^2\theta$, i.e.,
$x = \frac{sec^2\theta}{\sec^2\theta - 1} = \frac{sec^2\theta}{\tan^2\theta} = \frac{1}{\sin^2\theta}$. Hence $f(\sec^2\theta) = \sin^2\theta$.

11. The sum of all real numbers x such that $(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3$ is
(a) 3 **(b)** 7/2 **(c)** 2 **(d)** 5/2 **(e)** none of the above

The answer is: **(b)**
Solution:
Note: We shall use the identity $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ with $a = 2^x - 4$ and $b = 4^x - 2$, in which case, $a + b = 4^x + 2^x - 6$ and therefore we are given $(a + b)^3 = a^3 + b^3$. Hence $3a^2b + 3ab^2 = 0$, i.e., $a = 0$ or $b = 0$, or $a + b = 0$ after factoring by ab. Thus
$a = 2^x - 4 = 0$, i.e., $x = 2$;
or $b = 4^x - 2 = 0$, i.e., $x = 1/2$;
or $a + b = 4^x + 2^x - 6 = (2^x + 3)(2^x - 2) = 0$, i.e., $x = 1$.
Note that $2^x + 3 = 0$ has no real roots. Therefore, the sum of the roots is $2 + \frac{1}{2} + 1 = \frac{7}{2}$.

12. Let C be a point on the segment AB. Consider the region shaded below, that is bounded by the three semicircles with diameters AC, AB and BC respectively. Let a, b and c be the lengths of AC, CB and CD, respectively. Then the area of the region is



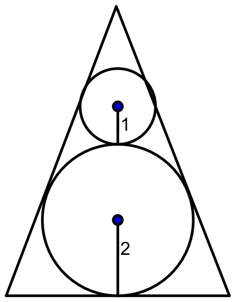
(a) $\frac{\pi}{2}ab$ cm^2 **(b)** $\frac{\pi}{4}ac$ cm^2 **(c)** $\frac{\pi}{4}bc$ cm^2 **(d)** $\frac{\pi}{4}c^2$ cm^2
(e) none of the above

The answer is: **(d)**

Solution:

In the first step we only use that the area of a disc of radius r is πr^2. Therefore, the sought area is
$A = \frac{\pi}{2}\left(\left(\frac{a+b}{2}\right)^2 - \left(\frac{a}{2}\right)^2 - \left(\frac{b}{2}\right)^2\right) = \frac{\pi}{4}ab.$
In the next step we use the properties of intersecting chords (or the formula for the height of the right triangle drawn from the hypothenuse to the opposite vertex) from which we have $c^2 = ab$ using that c is half of the length of the chord through C and D. Thus the above formula for the area can be written as $A = \frac{\pi}{4}c^2$.

13. In an isosceles triangle, the inscribed circle has radius 2. Another circle of radius 1 is tangent to the inscribed circle and the two equal sides. What is the area of the triangle?



(a) 20 **(b)** $11\sqrt{3}$ **(c)** $13\sqrt{2}$ **(d)** $16\sqrt{2}$ **(e)** none of the above

The answer is: **(d)**

Solution: Let A, B, and C be the vertices of the triangle with $AB = AC$. Let D be the center of the smaller circle and E the center of the larger circle. Let F be such that AF is an altitude for $\triangle ABC$. Let $h = AF$ and $x = FC$ so that the area of $\triangle ABC$ is hx. Drawing radii for the circles from their centers to the side AC, we obtain two smaller right triangles each having its hypotenuse on AF and each similar to $\triangle AFC$. Thus, $AD/1 = (AD + 3)/2 = AC/FC = \sqrt{h^2 + x^2}/x$. The first of these equations implies $AD = 3$. Since $h = AD + 5$, we obtain $h = 8$. Hence, $3 = \sqrt{64 + x^2}/x$ so that $9x^2 = 64 + x^2$. Thus, $x^2 = 8$ and $hx = 8\sqrt{8} = 16\sqrt{2}$.

14. Find the length of the third side of a triangle if the area of the triangle is 20 and two of its sides have lengths of 5 and 10.

(a) $\sqrt{65}$ **(b)** $3\sqrt{5}$ **(c)** 8 **(d)** 6 **(e)** none of the above

The answer is: **(a)**

Solution: Let ABC be the triangle with sides $AB = 5$ and $AC = 10$. The formula for the area using two sides AB and AC and the internal angle $\widehat{A}$ they make, may be written as follows $20 = (1/2) \times 5 \times 10 \times \sin(\widehat{A})$, i.e., $\sin(\widehat{A}) = 20/25 = 4/5$.
We now use the Cosine Law to find the length BC of the side opposing angle $\widehat{A}$, that is,
$BC^2 = 5^2 + 10^2 - 2 \times 5 \times 10 \times \cos(\widehat{A})$ with $\cos(\widehat{A}) = \sqrt{1 - \sin^2(\widehat{A})}$ to get
$BC^2 = 125 - 100\sqrt{1 - (4/5)^2} = 125 - 100(3/5) = 65$ and so $BC = \sqrt{65}$.

15. Nathan just aced his math test and he is hoping that his parents will reward him for his performance. Nathans parents decide that Nathan deserves a reward for his hard work; however, they like to add a little bit of chance to the reward. Nathans parents have 5 crisp new 5 dollar bills and 5 crisp new 10 dollar bills. They tell Nathan that he has to divide the bills into two groups. Nathans parents explain that after blindfolding Nathan they will place each group into a brown bag, after shuffling the bills. Then they will place one bag on the right hand side of a table and one on the left hand side of the table. He will choose one of the bags without examining them and then he will reach in and grab one of the bills. What is the highest probability of picking a 10 dollar bill that Nathan can achieve among all possible groupings of the bills?

(a) 4/9 **(b)** 5/9 **(c)** 13/18 **(d)** 1 **(e)** none of the above

The answer is: **(c)**
Solution: If Nathan divides the bills so that one \$10 bill is alone and the remaining bills are all together in the other bag, he has a 50% chance of choosing the bag with the \$10 bill in it and 50% chance of choosing the other bag. If he chooses the bag with the only \$10 bill Nathan will get a \$10 bill with a probability $p_1 = 1$, while if he chooses the other bag, the probability of choosing a \$10 bill is $p_2 = 4/9$. Thus, the probability of choosing a \$10 bill with this distribution of the bills is

$$\frac{1}{2}p_1 + \frac{1}{2}p_2 = \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{4}{9} = \frac{13}{18}.$$

We can always assume $p_1 > p_2$ (by letting bag 1 be the one with the highest proportion of ten dollar bills). In that case, any other combination will yield a lower probability since it will result in $p_1 \leq 1$ and $p_2 < 4/9$.

16. How many ways are there to choose 4 different numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ so that no two of the 4 numbers are consecutive?

(a) 20 **(b)** 35 **(c)** 40 **(d)** 45 **(e)** none of the above

The answer is: **(b)**

Solution: The problem is the same as counting the number of different strings of six 0's and four 1's with no two 1's appearing next to each other. For example, the string 0100100101 corresponds to choosing the numbers 2, 5, 8, and 10 in the problem (the numbers correspond to the positions of the 1's in the string of 0's and 1's). Such strings of 0's and 1's can be obtained by putting 1's in any 4 of the blanks below and bringing the 0's and 1's together.

$$\underset{1}{\_}\ 0\ \underset{2}{\_}\ 0\ \underset{3}{\_}\ 0\ \underset{4}{\_}\ 0\ \underset{5}{\_}\ 0\ \underset{6}{\_}\ 0\ \underset{7}{\_}$$

For example, the string 0100100101 above would correspond to putting 1's in the blanks numbered 2, 4, 6, and 7. There are 7 blanks and 4 are to be selected so that the total number of strings as above is $\begin{pmatrix} 7 \\ 4 \end{pmatrix} = \frac{7!}{3!4!} = 35$.

17. If $x, y > 0$, $\log_y x + \log_x y = \frac{10}{3}$ and $xy = 144$, then $\frac{x+y}{2} =$

(a) 30 **(b)** $12\sqrt{2}$ **(c)** $13\sqrt{3}$ **(d)** 24 **(e)** none of the above

The answer is: **(c)**

Solution: By symmetry in x and y, we can assume that $x > y$, and therefore that $\log_y x \geq 1$. Let $a = \log_y x$. Then, since $\log_x y = 1/a$, we solve $a + \frac{1}{a} = \frac{10}{3}$ to find $a = 3$ or $a = 1/3$. Since $a = \log_y x \geq 1$, $a = 3$, so $\log_y x = 3$, i.e., $x = y^3$. Now from $144 = xy = y^3 y = y^4$, it follows that $y = \sqrt[4]{144} = \sqrt{12} = 2\sqrt{3}$ and $x = y^3 = 24\sqrt{3}$. Thus

$$\frac{x+y}{2} = \frac{24\sqrt{3} + 2\sqrt{3}}{2} = 13\sqrt{3}.$$

18. How many real numbers x satisfy the following equation $\sqrt{3 + \sqrt{3 + \sqrt{3 + x}}} = x$?

(a) 0 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** infinitely many

The answer is: **(b)**

Solution: If $x < 0$ the equation has no solution. So, let $x \geq 0$. If $\sqrt{3+x} > x$, then $3 + \sqrt{3+x} > 3 + x$, and $\sqrt{3 + \sqrt{3+x}} > \sqrt{3+x} > x$. Add 3 to the outer parts of the last inequality and take square root to get $\sqrt{3 + \sqrt{3 + \sqrt{3+x}}} > \sqrt{3+x} > x$. So, the equation has no solution if $\sqrt{3+x} > x$. Similarly, if $\sqrt{3+x} < x$, the equation has no solution. Also, if $\sqrt{3+x} = x$ and $x \geq 0$, then there is only one solution. In fact, the quadratic equation $3 + x = x^2$ has one positive and one negative solution. The positive solution of $3 + x = x^2$ is the only solution of the equation of the problem.

19. What is the coefficient of x^{18} in the polynomial

$$(1+x)^{20}+x(1+x)^{19}+x^2(1+x)^{18}+\ldots+x^{18}(1+x)^2?$$

(a) 1310 **(b)** 1320 **(c)** 1330 **(d)** 1340 **(e)** none of the above

The answer is: **(c)**

Solution: Let

$$f(x)=(1+x)^{20}+x(1+x)^{19}+x^2(1+x)^{18}+\ldots+x^{18}(1+x)^2.$$

The polynomial $f(x)$ does not change if we multiply it by $1=(1+x)-x$. Thus,

$$\begin{aligned} f(x) &= ((1+x)-x)((1+x)^{20}+x(1+x)^{19}+x^2(1+x)^{18}+\ldots+x^{18}(1+x)^2) \\ &= (x+1)^{21}-x^{19}(1+x)^2. \end{aligned}$$

The coefficient of x^{18} when the expression $x^{19}(1+x)^2$ is expanded is clearly 0 (there is no x^{18} term), so the coefficient of x^{18} in $f(x)$ is simply the coefficient of x^{18} in $(1+x)^{21}$. By the binomial theorem, this is

$$\binom{21}{18}=\frac{21\times 20\times 19}{6}=7\times 10\times 19=1330.$$

20. Let $f(x)$ be a function such that $f(x)+f(\frac{1}{1-x})=x$ for all x not equal to 0 or 1. What is the value of $f(2)$?

(a) $\frac{3}{4}$ **(b)** $\frac{1}{4}$ **(c)** $\frac{7}{4}$ **(d)** Cannot be determined from the given information
(e) none of the above

The answer is: **(c)**

Solution: Substitute $x=2$ to get $f(2)+f(-1)=2$. Next, substitute $x=-1$ to get $f(-1)+f(1/2)=-1$. Finally, substitute $x=1/2$ to get $f(1/2)+f(2)=1/2$. Solving the above equations for $f(2)$ we obtain $f(2)=7/4$.

21. Let a, b, and c be the three roots of $x^3-64x-14$. What is the value of $a^3+b^3+c^3$?

(a) -36 **(b)** 42 **(c)** 12 **(d)** 36 **(e)** none of the above

The answer is: **(b)**

Solution: The hypotheses in the problem imply

$$x^3 - 64x - 14 = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + \ldots - abc.$$

Hence, $a+b+c=0$. Also, since a, b, and c are roots of $x^3 - 64x - 14$, we obtain $a^3 = 64a + 14$; $b^3 = 64b + 14$; and $c^3 = 64c + 14$.
Thus,

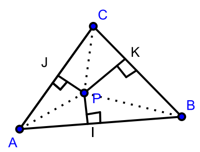
$$a^3 + b^3 + c^3 = 64(a+b+c) + 3 \times 14 = 64 \times 0 + 3 \times 14 = 42.$$

22. Let $\triangle ABC$ be an equilateral triangle whose side is of length 1 inch. Let P be a point inside the triangle $\triangle ABC$. Find the sum of the distances of P to the sides of the triangle $\triangle ABC$.

(a) $\frac{\sqrt{3}}{2}$ **(b)** $\frac{\sqrt{2}}{3}$ **(c)** $2\sqrt{3}$ **(d)** $\frac{\sqrt{2}}{2}$ **(e)** none of the above

The answer is: **(a)**

Solution



Let d_1, d_2 and d_3 be the distances of P to the sides of the triangle. Connect P with each of the vertices of the given triangle, which splits the given triangle into three smaller triangles $\triangle PBC$, $\triangle PAC$ and $\triangle PAB$.

The area S of the given triangle is the sum of the areas of the three small triangles. The formula for the area we want to use is that the area of a triangle is given by $\frac{1}{2}$ times the product of the length of any of its sides and the distance to it from the opposite vertex. Therefore, we have $S = \frac{1}{2}(d_1 + d_2 + d_3)$, since the side of the triangle is 1. This shows that the sought sum $d_1 + d_2 + d_3$ is independent of the position of P inside the triangle, and equals twice the area. If we take P to be one of the vertices it follows $d_1 + d_2 + d_3 = h$, where h is the height of the equilateral triangle. This also follows from $S = \frac{1}{2}h$. In any case, we have to find h, which can be done in many ways. For example by the Pythagorean theorem we have $h^2 + (1/2)^2 = 1$, which gives $h = \frac{\sqrt{3}}{2}$.

23. For how many integers n between 1 and 100 does $x^2 + x - n$ factor into the product of two linear factors with integer coefficients?

(a) 9 **(b)** 2 **(c)** 1 **(d)** 10 **(e)** none of the above

The answer is: **(a)**

Solution 1: Since $-n < 0$, it follows that $x^2 + x - n = (x - a)(x + b)$ where a and b are positive integers. Thus $n = ab$ and $b = 1 + a$ by expanding the right hand side of the above equality. Here are the possible choices for a and b:

a: 1 — 2 —. . .— 9 — 10

b: 2 — 3 —. . .— 10 — 11

$n = ab$: $1 \cdot 2 = 2$ — $2 \cdot 3 = 6$— . . . —$9 \cdot 10 = 90$ — $10 \cdot 11 = 110$.

There are 9 values for n between 1 and 100.

Solution 2: By the quadratic formula, $x^2 + x - n = (x - x_1)(x - x_2)$ where

$$x_1, x_2 = \frac{-1 \pm \sqrt{1 + 4n}}{2}.$$

Since x_1 and x_2 are integers, $1 + 4n = m^2$ for some positive integer m. Since $1 + 4n$ is odd, m must be odd, i.e., $m = 2k + 1$ for some integer k. Every odd square is of the form $1 + 4n$ since

$$m^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1.$$

Since $1 \leq n \leq 100$, the answer is the number of odd squares between $5 = 1 + 4 \cdot 1$ and $401 = 1+4\cdot 100$. There are nine odd squares in that range, $3^2, 5^2, 7^2, 9^2, 11^2, 13^2, 15^2, 17^2, 19^2$.

24. Two intersecting circles each have radius 6, and the distance between the centers of the circles is $6\sqrt{3}$. Find the area of the region that lies inside both circles.

(a) $12\pi - 24\sqrt{3}$ **(b)** $6\pi - 4\sqrt{3}$ **(c)** $12\pi - 18\sqrt{3}$ **(d)** $6\pi - 12\sqrt{3}$

(e) none of the above

The answer is: **(c)**

Solution: The area of one-fourth of the region can be seen to be the difference between the area of a sector with radius 6 and internal angle $\pi/6$ radians and the area of a right triangle with one side $3\sqrt{3}$ and hypotenuse 6. The area of the sector is $(1/2)(\pi/6)6^2 = 3\pi$. The area of the triangle is $(1/2)(3\sqrt{3})3 = 9\sqrt{3}/2$. Hence, the answer is $4\times(6\pi - 9\sqrt{3})/2 = 12\pi - 18\sqrt{3}$.

25. Suppose $0 \leq a_i < n$ for $i = 0, 1, 2, \ldots, r$. The number $(a_r a_{r-1} \ldots a_1 a_0)_n$ represents the number $a_r n^r + \ldots + a_1 n + a_0$ in base n. For example, $(102)_{13}$ is the base 13 representation of $1 \cdot 13^2 + 0 \cdot 13^1 + 2 \cdot 13^0 = 13^2 + 2 = 171$. In which bases n is $(11)_n$ a perfect square?

(a) when n is a perfect square
(b) for all positive integer n
(c) no such n exists
(d) when n is one less than a perfect square
(e) none of the above

The answer is: **(d)**

Solution: We want $(11)_n = n + 1 = m^2$, i.e., $n = m^2 - 1$. Note: n and m are positive integers, $m \geq 2$. So $n = \{m^2 - 1 | m = 2, 3, 4, \ldots\} = \{3, 8, 15, 24, \ldots\}$.

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Dear Teachers/Students:

If you do have any suggestions about the competition, or if you have different solutions to any of this year's problems, please send them by mail or e-mail to

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Remember to visit us for information about past competitions at
http://www.svsu.edu/matholympics/

The SVSU Math Olympic Committee would like to express his gratitude to all participants.

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