SAGINAW VALLEY STATE UNIVERSITY
2013 MATH OLYMPICS LEVEL II

1. The following inequalities hold for all positive integers n:

$$\sqrt{n+1}-\sqrt{n}<\frac{1}{\sqrt{4n+1}}<\sqrt{n}-\sqrt{n-1}.$$

What is the greatest integer which is less than

$$\sum_{n=1}^{24}\frac{1}{\sqrt{4n+1}}?$$

(a) 2 **(b)** 3 **(c)** 4 **(d)** 5 **(e)** 6

2. Find the closest integer to $a+b$ if

$$a=\frac{1}{4}+\frac{3}{8}+\frac{5}{12}+\ldots+\frac{1005}{2012}$$

and

$$b=\frac{5}{8}+\frac{7}{12}+\frac{9}{16}+\ldots+\frac{1009}{2016}.$$

(a) 503 **(b)** 502 **(c)** 1 **(d)** 0 **(e)** none of the above

3. How many 4 digit numbers with first digit 2 have exactly one pair of two identical digits (like 2011, or 2012)?

(a) 216 **(b)** 108 **(c)** 432 **(d)** 54 **(e)** none of the above

4. Which of the following is an algebraic expression of $\sin(\sin^{-1}x+\cos^{-1}y)$, for $-1\le x\le 1$ and $-1\le y\le 1$?

(a) $xy+\sqrt{1-x^2}\sqrt{1-y^2}$ **(b)** $y\sqrt{1-x^2}+x\sqrt{1-y^2}$ **(c)** $2\sqrt{1-x^2}\sqrt{1-y^2}$
(d) $xy\pm\sqrt{1-x^2}\sqrt{1-y^2}$ **(e)** none of the above

5. The Bayes family has two cars. One of them has a 90% chance of starting on any given morning, and the other has an 80% chance. On any given morning, what is the probability that at least one of the cars will start?

(a) .72 **(b)** .9 **(c)** .7 **(d)** .98 **(e)** none of the above

6. In the sum below, the letter $F = 0$, and the other letters represent the digits $1, 2, 3, 4, 5$, or 6, with each digit used exactly once. The 2-digit integer AB is a prime number. What is the value of $A + B$?

$$\begin{array}{r} AB \\ +\ \ CD \\ \hline EFG \end{array}$$

(a) 7 **(b)** 5 **(c)** 4 **(d)** 3 **(e)** none of the above

7. Which of the following functions is the inverse to the function
$f(x) = x^{15} - 6x^{10} + 12x^5 - 8$?

(a) No inverse exists because the function is not one-to-one **(b)** $f^{-1}(x) = \sqrt[15]{\frac{x+8}{x^{15}+6x^{10}+12}}$

(c) $f^{-1}(x) = \sqrt[5]{\sqrt[3]{x} + 2}$ **(d)** $f^{-1}(x) = \sqrt[15]{\frac{\sqrt[10]{x}}{6} + \frac{\sqrt[5]{x}}{12} + 8}$

(e) none of the above

8. A finite sequence $a_0, a_1, \ldots, a_n$ of integers is called a *curious sequence* if it has the property that for every $k = 0, 1, 2, \ldots, n$, the number of times k appears in the sequence is a_k. For example, $a_0 = 1$, $a_1 = 2, a_2 = 1, a_3 = 0$ forms a curious sequence. Let $a_0, a_1, \ldots, a_{100}$ be a curious sequence. What is the value of the sum $\sum_{k=0}^{100} a_k$?

(a) 201 **(b)** 101 **(c)** 200 **(d)** 100 **(e)** none of the above

9. $\sin 3\theta = ?$

(a) $3(\cos^2\theta - \sin^2\theta)$ **(b)** $\sin^2\theta\cos\theta$ **(c)** $3\sin\theta$ **(d)** $3\sin\theta - 4\sin^3\theta$
(e) none of the above

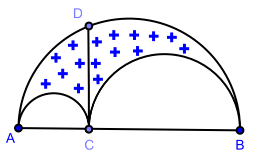
10. If $f\left(\frac{x}{x-1}\right) = \frac{1}{x}$ for all $x \neq 0, 1$ and $0 < \theta < \pi/2$, then $f(\sec^2\theta) =$

(a) $\cot^2\theta$ **(b)** $\tan^2\theta$ **(c)** $\cos^2\theta$ **(d)** $\sin^2\theta$
(e) none of the above

11. The sum of all real numbers x such that $(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3$ is

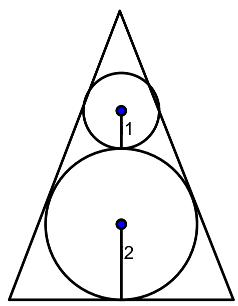
(a) 3 **(b)** $7/2$ **(c)** 2 **(d)** $5/2$ **(e)** none of the above

12. Let C be a point on the segment AB. Consider the region shaded below, that is bounded by the three semicircles with diameters AC, AB and BC respectively. Let a, b and c be the lengths of AC, CB and CD, respectively. Then the area of the region is



(a) $\frac{\pi}{2}ab$ cm^2 **(b)** $\frac{\pi}{4}ac$ cm^2 **(c)** $\frac{\pi}{4}bc$ cm^2 **(d)** $\frac{\pi}{4}c^2$ cm^2
(e) none of the above

13. In an isosceles triangle, the inscribed circle has radius 2. Another circle of radius 1 is tangent to the inscribed circle and the two equal sides. What is the area of the triangle?



(a) 20 **(b)** $11\sqrt{3}$ **(c)** $13\sqrt{2}$ **(d)** $16\sqrt{2}$ **(e)** none of the above

14. Find the length of the third side of a triangle if the area of the triangle is 20 and two of its sides have lengths of 5 and 10.

(a) $\sqrt{65}$ **(b)** $3\sqrt{5}$ **(c)** 8 **(d)** 6 **(e)** none of the above

15. Nathan just aced his math test and he is hoping that his parents will reward him for his performance. Nathans parents decide that Nathan deserves a reward for his hard work; however, they like to add a little bit of chance to the reward. Nathans parents have 5 crisp new 5 dollar bills and 5 crisp new 10 dollar bills. They tell Nathan that he has to divide the bills into two groups. Nathans parents explain that after blindfolding Nathan they will place each group into a brown bag, after shuffling the bills. Then they will place one bag on the right hand side of a table and one on the left hand side of the table. He will choose one of the bags without examining them and then he will reach in and grab one of the bills. What is the highest probability of picking a 10 dollar bill that Nathan can achieve among all possible groupings of the bills?

(a) 4/9 **(b)** 5/9 **(c)** 13/18 **(d)** 1 **(e)** none of the above

16. How many ways are there to choose 4 different numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ so that no two of the 4 numbers are consecutive?

(a) 20 **(b)** 35 **(c)** 40 **(d)** 45 **(e)** none of the above

17. If $x, y > 0$, $\log_y x + \log_x y = \frac{10}{3}$ and $xy = 144$, then $\frac{x+y}{2} =$

(a) 30 **(b)** $12\sqrt{2}$ **(c)** $13\sqrt{3}$ **(d)** 24 **(e)** none of the above

18. How many real numbers x satisfy the following equation $\sqrt{3 + \sqrt{3 + \sqrt{3 + x}}} = x$?

(a) 0 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** infinitely many

19. What is the coefficient of x^{18} in the polynomial

$$(1 + x)^{20} + x(1 + x)^{19} + x^2(1 + x)^{18} + \ldots + x^{18}(1 + x)^2?$$

(a) 1310 **(b)** 1320 **(c)** 1330 **(d)** 1340 **(e)** none of the above

20. Let $f(x)$ be a function such that $f(x) + f(\frac{1}{1-x}) = x$ for all x not equal to 0 or 1. What is the value of $f(2)$?

(a) $\frac{3}{4}$ **(b)** $\frac{1}{4}$ **(c)** $\frac{7}{4}$ **(d)** Cannot be determined from the given information
(e) none of the above

21. Let a, b, and c be the three roots of $x^3 - 64x - 14$. What is the value of $a^3 + b^3 + c^3$?

(a) -36 **(b)** 42 **(c)** 12 **(d)** 36 **(e)** none of the above

22. Let $\triangle ABC$ be an equilateral triangle whose side is of length 1 inch. Let P be a point inside the triangle $\triangle ABC$. Find the sum of the distances of P to the sides of the triangle $\triangle ABC$.

(a) $\frac{\sqrt{3}}{2}$ **(b)** $\frac{\sqrt{2}}{3}$ **(c)** $2\sqrt{3}$ **(d)** $\frac{\sqrt{2}}{2}$ **(e)** none of the above

23. For how many integers n between 1 and 100 does $x^2 + x - n$ factor into the product of two linear factors with integer coefficients?

(a) 9 **(b)** 2 **(c)** 1 **(d)** 10 **(e)** none of the above

24. Two intersecting circles each have radius 6, and the distance between the centers of the circles is $6\sqrt{3}$. Find the area of the region that lies inside both circles.

(a) $12\pi - 24\sqrt{3}$ **(b)** $6\pi - 4\sqrt{3}$ **(c)** $12\pi - 18\sqrt{3}$ **(d)** $6\pi - 12\sqrt{3}$
(e) none of the above

25. Suppose $0 \leq a_i < n$ for $i = 0, 1, 2, \ldots, r$. The number $(a_r a_{r-1} \ldots a_1 a_0)_n$ represents the number $a_r n^r + \ldots + a_1 n + a_0$ in base n. For example, $(102)_{13}$ is the base 13 representation of $1 \cdot 13^2 + 0 \cdot 13^1 + 2 \cdot 13^0 = 13^2 + 2 = 171$. In which bases n is $(11)_n$ a perfect square?

(a) when n is a perfect square **(b)** for all positive integer n
(c) no such n exists **(d)** when n is one less than a perfect square
(e) none of the above