SOLUTIONS OF 2012 MATH OLYMPICS LEVEL II

1. If $T_n = 1 + 2 + 3 + \ldots + n$ and

$$P_n = \frac{T_2}{T_2 - 1} \cdot \frac{T_3}{T_3 - 1} \cdot \frac{T_4}{T_4 - 1} \cdot \ \cdots \ \cdot \frac{T_n}{T_n - 1}.$$

for $n = 2, 3, 4, \ldots$, then P_{2012} is the closest to which of the following numbers?

(a) 2.9 **(b)** 2.3 **(c)** 3.1 **(d)** 2.6 **(e)** 3.5

The answer is: **(a)**

Solution: Note that $T_n = \frac{n(n+1)}{2}$. So

$$P_n = \frac{T_n}{T_n - 1} P_{n-1} = \frac{n(n+2)/2}{[n(n+1) - 2]/2} P_{n-1} = \frac{n(n+1)}{(n+2)(n-1)} P_{n-1}.$$

Thus,

$$\begin{aligned} P_n &= \prod_{i=2}^{n} \frac{i(i+1)}{(i+2)(i-1)} \\ &= \frac{\left(\prod_{i=2}^{n} i\right)\left(\prod_{i=2}^{n}(i+1)\right)}{\left(\prod_{i=2}^{n}(i+2)\right)\left(\prod_{i=2}^{n}(i-1)\right)} \\ & \frac{n!\left(\frac{(n+1)!}{2}\right)}{\left(\frac{(n+2)!}{2\cdot 3}\right)(n-1)!} \\ &= \frac{3n}{n+2}. \end{aligned}$$

Hence $P_{2012} = (3 \cdot 2012)/2014$ which is closest to 2.9. In fact, without computation, on can use the fact that the limit of the sequence P_n is 3.

2. Find the value of the sum $\displaystyle\sum_{k=1}^{100} \frac{1}{k(k+1)}$.

(a) $\frac{99}{100}$ **(b)** $\frac{1}{10100}$ **(c)** $\frac{100}{101}$ **(d)** $\frac{194}{100}$ **(e)** none of the above

The answer is: **(c)**

Solution: First note that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. So $\displaystyle\sum_{k=1}^{100} \frac{1}{k(k+1)} = \sum_{k=1}^{100} \left(\frac{1}{k} - \frac{1}{k+1}\right)$. By expanding the sum, the central negative terms cancelled out one term on the left and another on the right, except the edges where one of them is missing. This is an example of a telescopic sum. Cancelling everything there is to be cancelled, we are left with, $\displaystyle\sum_{k=1}^{100} \frac{1}{k(k+1)} = 1 - \frac{1}{101} = \frac{100}{101}$.

3. What is the last digit of 3^{2012}?

(a) 1 **(b)** 3 **(c)** 7 **(d)** 9 **(e)** none of the above

The answer is: **(a)**

Solution: The last digit of 3^n follows a pattern: $3, 9, 7, 1, 3, 9, 7, 1, \ldots$ The 2012^{th} element of this sequence is 1.

4. Which of the following is equivalent to $\tan(\frac{1}{2}\cos^{-1} x)$, for $-1 < x \leq 1$?

(a) $\frac{\sqrt{2}}{\sqrt{x^2+4}}$ **(b)** $\frac{\sqrt{1-x^2}}{1+x}$ **(c)** $\frac{\sqrt{4-x^2}}{x}$ **(d)** $\pm\sqrt{\frac{1+x^2+\sqrt{1-x^2}}{2(1+x^2)}}$

(e) none of the above

The answer is: **(b)**

Solution: Let $\alpha = \cos^{-1} x$, i.e., $x = \cos\alpha$. We use the formula $\tan(\frac{\alpha}{2}) = \frac{\sin\alpha}{1+\cos\alpha}$ (if this formula is not remembered, one can get it from the half-angle formula for sine and cosine) to see that

$$\tan(\frac{1}{2}\cos^{-1} x) = \tan(\frac{\alpha}{2}) = \frac{\sin\alpha}{1+\cos\alpha} = \frac{\sqrt{1-x^2}}{1+x}.$$

5. Suppose a group of people have a code between themselves on how they can send messages to others in the group.

- For each subgroup of distinct people A, B and C in the group the following holds: if A can send a message to B and B can send a message to C, then C can send a message to A.
- For each pair of distinct people A and B in the group, either A can send a message to B or B can send a message to A but not both.

What is the largest number of people in the group?

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** none of the above

The answer is: **(c)**

Solution: Let X be the group of people. Sending of messages among the group translates to a relation R on X (a subset of $X \times X$) such that if $(A, B) \in R$ and $(B, C) \in R$, then $(C, A) \in R$ and either $(A, B) \in R$ or $(B, A) \in R$ but not both. A relation with these properties forces X to have at most 3 elements, since any relation with 4 or more elements will have both (A, B) and (B, A) for some A and B in X. Note that if $(A, B) \in R$ and $(B, C) \in R$ and $(B, D) \in R$, then $(C, A) \in R$ and $(D, A) \in R$. We only do not know the relationship between C and D. If $(C, D) \in R$, then $(D, B) \in R$ which cannot be the case since $(B, D) \in R$. If $(D, C) \in R$, then $(C, B) \in R$ which also cannot be the case.

6. A problem to remember the year 2011: how many positive integers less than or equal to 2011 are multiples of both 3 and 5 but not multiples of 8?

(a) 110 **(b)** 118 **(c)** none **(d)** 30 **(e)** none of the above

The answer is: **(b)**

Solution: Let M_k be the set of positive integers less than or equal to 2011 which are multiples of k. Let $\mid M_k \mid$ be the number of elements of the set M_k, i.e., the number of positive integers less than or equal to 2011 which are multiples of k. With this notation, the question is to find the number $|M_3 \cap M_5| - |M_3 \cap M_5 \cap M_8|$, where the symbol $\cap$ is used, as usual, for the intersection (the common elements) of the involved sets. Thus, for example, $M_3 \cap M_5$ is the set of positive integers less than or equal to 2011 which are multiples of both 3 and 5. A key observation, which we shall used next, is that if m and n are two relatively prime integer numbers, i.e., they do not have any common integer divisor besides ± 1, then any number which is divisible by both m and n is also divisible by mn. Therefore

$$|M_3 \cap M_5| - |M_3 \cap M_5 \cap M_8| = |M_{15}| - |M_{120}|.$$

Next, we use that $|M_k|$ is the number of times k "fits" inside 2011. In other words, $|M_k|$ is the largest positive number whose product with k is less than or equal to 2011, but whose product with $k + 1$ is greater than 2011. The greatest integer value function is defined by these two properties, so

$$|M_k| = \left[\frac{2011}{k}\right].$$

In particular $\left[\frac{2011}{15}\right] = 134$ and $\left[\frac{2011}{120}\right] = 16$ taking onto account that $134 < \frac{2011}{15} < 135$ while $16 < \frac{2011}{120} < 17$. Hence $|M_{15}| - |M_{120}| = 134 - 16 = 118$.

7. At an artisan bakery, French tortes are 52 dollars each, almond tarts are 12 dollars each and cookies are one dollar each. If Alex has 400 dollars to purchase exactly 100 of these items for a party and he buys at least one of each item, how many French Tortes does he purchase? Only whole pieces of the bakery items can be purchased.

(a) 4 **(b)** 5 **(c)** 3 **(d)** 1 **(e)** 2

The answer is: **(e)**

Solution: Solution: Suppose F is the number of French tortes, A is the number of almond tarts purchased and C is the number of cookies purchased. Then we have two equations:

$$52F + 12A + C = 400$$

$$F + A + C = 100.$$

Since at least one of each item is purchased then $1 \leq F, A, C$. Since the French tortes are the most expensive, we see that we cannot purchase more than 7 tortes or we will go over the 400 dollars we have to spend. By trial and error we see that we will not obtain integer solutions if $F \neq 2$. However, when $F = 2$, we obtain the two equations in two unknowns:

$$12A + C = 296$$

$$A + C = 98$$

which have solution $A = 18, C = 80$. Alternatively, we solve the system

$$12A + C = 400 - 52F$$

$$A + C = 100 - F$$

considering F as a parameter. The solution is

$$A = \frac{300}{11} - \frac{51}{11}F$$

$$C = \frac{40}{11}F + \frac{800}{11}.$$

In particular, F is 1, 2, 3 or 4 so that $A > 0$. Only for $F = 2$ we obtain a solution which is of whole numbers.

8. We are given 6 points on a circle equally spaced (think of a clock with just the even hours). How many triangles can be constructed so that their vertices are three of the given points on the circle? How many among those triangles are isosceles triangles (isosceles means the triangle has at least two equal sides)?

(a) 20 triangles, 6 isosceles **(b)** 18 triangles, 8 isosceles **(c)** 20 triangles, 8 isosceles
(d) 18 triangles, 6 isosceles **(e)** none of the above

The answer is: **(c)**

Solution: Since no three points are collinear, any choice of three will make the vertices of a triangle, so there are

$$\binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20, \text{ triangles.}$$

For the isosceles triangles, we pick a starting vertex, then pick the other two points to be an equal distance away, going both directions. There are two such pairs of points , for each starting point, however, one of the two pairs will form an equilateral triangle, which means that if we just take $6 \times 2 = 12$, we will be over-counting. There are $6 \times 1 = 6$ non-equilateral isosceles triangles and two equilateral triangles, for a total of 8 isosceles triangles.

9. $\cos 3\theta = ?$

(a) $4\cos^3\theta - 3\cos\theta$ **(b)** $2(\cos^2\theta - \sin^2\theta)$ **(c)** $\cos\theta(1 + 2\sin^2\theta)$
(d) $4\cos^3\theta - \cos\theta$ **(e)** none of the above

The answer is: **(a)**

Solution: It is easy to see that
$\cos 3\theta = \cos(2\theta + \theta) = cos2\theta\cos\theta - \sin 2\theta \sin\theta = (2\cos^2\theta - 1)\cos\theta - 2\cos\theta\sin\theta\sin\theta = 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta = 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta) = 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta = 4\cos^3\theta - 3\cos\theta$.

10. What is the value of the following product $\sin\frac{\pi}{32}\cos\frac{\pi}{32}\cos\frac{\pi}{16}\cos\frac{\pi}{8}\cos\frac{\pi}{4}$?

(a) $\frac{1}{2}$ **(b)** $\frac{1}{4}$ **(c)** $\frac{1}{8}$ **(d)** $\frac{1}{16}$ **(e)** none of the above

The answer is: **(d)**

Solution: Using the double angle formula, we have

$$\begin{aligned}
&\sin\tfrac{\pi}{32}\cos\tfrac{\pi}{32}\cos\tfrac{\pi}{16}\cos\tfrac{\pi}{8}\cos\tfrac{\pi}{4} \\
&= \tfrac{1}{2}\sin\tfrac{\pi}{16}\cos\tfrac{\pi}{16}\cos\tfrac{\pi}{8}\cos\tfrac{\pi}{4} \\
&= \tfrac{1}{4}\sin\tfrac{\pi}{8}\cos\tfrac{\pi}{8}\cos\tfrac{\pi}{4} \\
&= \tfrac{1}{8}\sin\tfrac{\pi}{4}\cos\tfrac{\pi}{4} \\
&= \tfrac{1}{16}\sin\tfrac{\pi}{2} = \tfrac{1}{16}.
\end{aligned}$$

11. What is the remainder when $P(x) = 1 - x + 2x^4 - 3x^9 + 4x^{16} - 5x^{25} + x^{2011} + 6x^{2012}$ is divided by $D(x) = x^2 - 1$?

(a) 15 **(b)** $-2x + 3$ **(c)** $-8x + 13$ **(d)** $x^2 - x + 1$

(e) none of the above

The answer is: **(c)**

Solution:

Note: The remainder is a polynomial $R(x)$ of degree smaller than the divisor $D(x)$ such that there is another polynomial $Q(x)$, the quotient, such that $P(x) = Q(x)D(x) + R(x)$: Given a polynomial $P(x)$ and a divisor $D(x)$ then the remainder and the quotient are uniquely determined.

Since $D(x) = x^2 - 1$ is degree 2, the remainder polynomial must be of degree 1 or lower, so we may write it as $R(x) = ax + b$. Note that $D(x)$ has roots ± 1. Since $P(x) = Q(x)D(x) + R(x) = Q(x)D(x) + ax + b$, we know that $P(1) = Q(1)D(1) + a + b = Q(1) \cdot 0 + a + b = a + b$; because 1 is a root of D(x). Similarly, $P(-1) = Q(-1)D(-1) - a + b = 0 - a + b = -a + b$: Computing $P(1) = 5$ and $P(-1) = 21$ directly from the form of $P(x)$, we have

$$a + b = 5$$

$$-a + b = 21$$

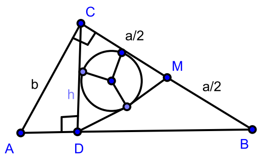
This is easily solved for $b = 13$ and $a = -8$. We have $R(x) = -8x + 13$.

12. Let $\triangle ABC$ be a right triangle ($\angle C = 90°$) such that $AB = 25$ cm, $BC > AC$ and the radius r of the inscribed circle in $\triangle ABC$ is 5 cm. Let also CD be the height of $\triangle ABC$ from the vertex C to the side AB. Find the radius of the circle inscribed in $\triangle CDM$ where M is the midpoint of the side BC.

(a) 3 cm **(b)** 2 cm **(c)** 4 cm **(d)** 5 cm **(e)** none of the above

The answer is: **(a)**

Solution:



$r = 5$ cm, $c = 25$ cm and note that $r = \frac{1}{2}(a + b - c)$. Thus $a + b = 35$.

The area of the triangle $\triangle ABC$ is $A_{\triangle ABC} = \frac{1}{2}r(a + b + c) = 150$. Since $A_{\triangle ABC} = \frac{1}{2}ab$, it follows that $ab = 300$. Since $a + b = 35$ and $ab = 300$, we have $a = 20$ and $b = 15$.

Now, let r_1 be the radius of the inscribed circle of $\triangle CDM$.

Then $A_{\triangle CDM} = \frac{1}{2}r_1(CD + CM + DM) = \frac{1}{2}r_1(h + \frac{a}{2} + \frac{a}{2}) = \frac{1}{2}r_1(h + a)$.

Note that $\sin \widehat{B} = \frac{h}{a} = \frac{b}{c}$. So $BD^2 = a^2 - h^2 = a^2 - (\frac{ab}{c})^2 = \frac{a^4}{c^2}$, i.e., $BD = \frac{a^2}{c}$. Thus

$A_{\triangle CDM} = \frac{1}{2}A_{\triangle BCD} = \frac{1}{2}\cdot\frac{1}{2}h\cdot BD = \frac{1}{2}\frac{a^2}{c^2}A_{\triangle ABC} = \frac{1}{2}\frac{20^2}{25^2}\cdot 150 = 48.$
Hence $\frac{1}{2}r_1(h+a) = 48$ and $r_1 = \frac{96}{a+h}$. Since $h = \frac{ab}{c} = 12$, we have $r_1 = \frac{96}{20+12} = 3$, i.e., $r_1 = 3$ cm.

13. The triangle $\triangle ABC$ has $AB = 7$ and the lengths of the other two sides have given ratio $BC/CA = 24/25$. What is the largest possible area for the $\triangle ABC$?
(a) $25\sqrt{2}$ **(b)** 300 **(c)** 12 **(d)** $5\sqrt{2}$ **(e)** none of the above

The answer is: **(b)**
Solution: Suppose that one has the vertices A, B and C of the constructed triangle be $(0,0), (-a,0)$ and (x,y) respectively. Note that $AB = a = 7$ is fixed. Note that if $b = 24$ and $c = 25$, then $a^2 + b^2 = c^2$. So one can assume $BC = kb$ and $AC = kc$ for some positive real number k, with $a^2 + b^2 = c^2$. So kc is the distance from (x,y) to $(-a,0)$ and kb is the distance from (x,y) to $(0,0)$. Thus,

$$\frac{(x+a)^2 + y^2}{x^2 + y^2} = \frac{k^2c^2}{k^2b^2} = \frac{c^2}{b^2}.$$

This gives

$$b^2[(x+a)^2 + y^2] = c^2[x^2 + y^2].$$

Expanding and moving some terms gives $b^2a^2 = (c^2 - b^2)x^2 - 2ab^2x + a^2y^2$, or $b^2a^2 = a^2x^2 - 2ab^2x + a^2y^2$. Dividing out by a^2 gives $b^2 = x^2 - \frac{2b^2}{a}x + y^2$. Now, completing the square for the x terms gives

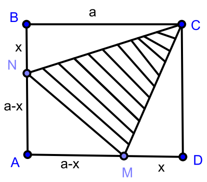
$$b^2 + \frac{b^4}{a^2} = (x - \frac{b^2}{a})^2 + y^2.$$

Simplifying with the Pythagorean Theorem again yields

$$\frac{b^2c^2}{a^2} = (x - \frac{b^2}{a})^2 + y^2.$$

This last is the sum of two squares, and y is clearly maximized when the square term with x is zero, giving $y_{max} = \frac{bc}{a}$.
But the area of the triangle ABC is $\mathcal{A} = \frac{1}{2}ay$, where y is the height of the triangle and a the fixed base. Thus, the maximum area of such a triangle is $\mathcal{A}_{max} = \frac{1}{2}ay_{max} = \frac{1}{2}a\cdot\frac{bc}{a} = \frac{1}{2}bc$. For our special case, $b = 24$ and $c = 25$, giving area 300.

14. An equilateral triangle is inscribed in a square with side a so that one of the vertices of the triangle coincides with one of the vertices of the square. Find the area of the triangle in terms of a.



(a) $(2\sqrt{2}-3)a^2$ **(b)** $(2\sqrt{3}-3)a^2$ **(c)** $(2\sqrt{5}-3)a^2$ **(d)** $(2\sqrt{5}-4)a^2$
(e) none of the above

The answer is: **(b)**

Solution 1: Let $ABCD$ be the square and MNC the triangle. Let $\mid BN \mid = x$. Then $\mid DM \mid = x$ and $\mid AN \mid = \mid AM \mid = a-x$. Let $\mid CN \mid = \mid CM \mid = \mid MN \mid = y$. From the right triangle $\triangle MAN$ we have $(a-x)^2+(a-x)^2 = y^2$ and from the right triangle $\triangle CBN$, we have $a^2+x^2 = y^2$. Thus $a^2+x^2 = 2(a-x)^2$ which is equivalent to $x^2-4ax+a^2 = 0$. The quadratic formula gives $x = \frac{4a\pm\sqrt{16a^2-4a^2}}{2} = \frac{4a\pm 2a\sqrt{3}}{2} = (2\pm\sqrt{3})a$. Since the solution $(2+\sqrt{3})a$ is greater than a, it follows that $x = (2-\sqrt{3})a$. Hence $y^2 = 2(a-x)^2 = 2(a-(2-\sqrt{3})a)^2 = 2(\sqrt{3}-1)^2a^2$. So the area of $\triangle MCN$ is $\mathcal{A} = \frac{\sqrt{3}}{4}y^2 = \frac{\sqrt{3}}{4}2(\sqrt{3}-1)^2a^2 = \frac{\sqrt{3}}{2}(4-2\sqrt{3})a^2 = (2\sqrt{3}-3)a^2$.

15. Horses A, B and C are entered in a three-horse race in which ties are not possible. If the odds against A winning are 3-to-1 and the odds against B winning are 2-to-3, what are the odds against C winning? (By "odds against X winning are p-to-q" we mean that the probability of X winning the race is $\frac{q}{p+q}$.)

(a) 3-to-20 **(b)** 5-to-6 **(c)** 8-to-5 **(d)** 17-to-3 **(e)** 20-to-3

The answer is: **(d)**

Solution: The probability that A wins is 1/(3+1) and the probability that B wins is 3/(2+3). The sum of the winning probabilities for all the three horses is 1, so the probability that C wins is

$$1-\frac{1}{4}-\frac{3}{5}=\frac{3}{20}=\frac{3}{17+3}.$$

Hence the odds against C winning are 17-to-3.

16. A frog makes 2 jumps, each 1 meter in length. The direction of the jumps are chosen independently and at random in any direction in a plane. What is the probability that the frogs final position is at most 1 meter from its starting position?

(a) $\frac{1}{2}$ **(b)** $\frac{2}{3}$ **(c)** $\frac{1}{3}$ **(d)** $\frac{1}{4}$ **(e)** none of the above

The answer is: **(c)**

Solution: Set up a Cartesian coordinate system for a frog's position. Suppose the frog starts at $(1, 0)$ and his first jump is along the x-axis and puts him at $(0, 0)$. Then his second jump must be at angle θ where $-60° \leq \theta \leq 60°$ in order for the distance between his final position and his original position to be less than 1. This range of angles is one third of the circle, hence the probability is one third.

17. Let z_1, z_2 and z_3 be the roots of the polynomial $Q(x) = x^3 - 9x^2 + 1$. In other words, $Q(z_1) = Q(z_2) = Q(z_3) = 0$. If $P(x) = x^5 - x^2 - x$, what is the value of $P(z_1)+P(z_2)+P(z_3)$?

(a) 0 **(b)** $153,300$ **(c)** $13,320$ **(d)** $58,554$ **(e)** none of the above

The answer is: **(d)**

Solution: Since z_1, z_2 and z_3 are the roots of Q, $z_i^3 = 9z_i^2 - 1$ for each $i = 1, 2, 3$. Thus $z_i^5 = 9z_i^4 - z_i^2 = 9z_i(9z_i^2 - 1) - z_i^2 = 9^2 z_i^3 - z_i^2 - 9z_i = 9^2(9z_i^2 - 1) - z_i^2 - 9z_i = (9^3 - 1)z_i^2 - 9z_i - 9^2$. Hence, $P(z_i) = z_i^5 - z_i^2 - z_i = (9^3 - 1)z_i^2 - 9z_i - 9^2 - z_i^2 - z_i = (9^3 - 2)z_i^2 - 10z_i - 9^2$.
On the other hand, by Vieta's formulas, we have

$$9 = z_1 + z_2 + z_3, 0 = z_1 z_2 + z_1 z_3 + z_2 z_3, -1 = z_1 z_2 z_3.$$

Therefore

$$\begin{aligned}
P(z_1) + P(z_2) + P(z_3) &= (9^3 - 2)(z_1^2 + z_2^2 + z_3^2) - 10(z_1 + z_2 + z_3) - 3 \cdot 9^2 \\
&= (9^3 - 2)((z_1 + z_2 + z_3)^2 - 2(z_1 z_2 + z_1 z_3 + z_2 z_3)) - 9 \cdot 10 - 3 \cdot 9^2 \\
&= (9^3 - 2) \cdot 9^2 - 9 \cdot 10 - 3 \cdot 9^2 \\
&= 9^2 \cdot (9^3 - 5) - 9 \cdot 10 \\
&= 58,554
\end{aligned}$$

18. $\frac{\frac{1}{5}+1}{\frac{1+\sqrt[3]{5}}{5}}$ is equal to which of the following numbers?

(a) $\sqrt[3]{5} - 1$ **(b)** $6(1 + \sqrt[3]{5})$ **(c)** $\frac{-3(1+\sqrt[3]{5})}{2}$ **(d)** $1 - \sqrt[3]{5} + \sqrt[3]{25}$
(e) none of the above

The answer is: **(d)**

Solution:

$$\frac{\frac{1}{5}+1}{\frac{1+\sqrt[3]{5}}{5}} = \frac{(\frac{1}{5}+1)\cdot 5}{(\frac{1+\sqrt[3]{5}}{5})\cdot 5} = \frac{1+5}{1+\sqrt[3]{5}} = \frac{6(1-\sqrt[3]{5}+\sqrt[3]{25})}{(1+\sqrt{5})(1-\sqrt[3]{5}+\sqrt[3]{25})} = \frac{6(1-\sqrt[3]{5}+\sqrt[3]{25})}{1^3+(\sqrt[3]{5})^3} = 1-\sqrt[3]{5}+\sqrt[3]{25}.$$

19. For each real number x , let $g(x)$ be the minimum value of the numbers $6x+3, 2x+7, 15-x$. (For example if $x=2$, then the three numbers are 15,11,13, so $g(2)=11$.) What is the maximum value of $g(x)$?

(a) $\frac{125}{9}$ **(b)** $\frac{40}{3}$ **(c)** $\frac{109}{9}$ **(d)** $\frac{37}{3}$ **(e)** none of the above

The answer is: **(d)**

Solution: Notice that the graph of g is obtained by taking the lowest of the graphs of the given functions. In our case, if we graph the lines $y=6x+3, y=2x+7$ and $y=-x+15$, then the graph of g is the boundary of the region which is below each of the graphs. The graphs of the lines $y=6x+3, y=2x+7$ and $y=-x+15$ have three points of intersection, which are at $x=1, x=\frac{12}{7}$ and $x=\frac{8}{3}$. For example, the first is obtained by solving $6x+3=2x+7$, i.e., $4x=4$ or $x=1$ (and $y=9$ of this common point).

Comparing the values of $g(x)$ at the three intersection points of the lines, where $g(x)$ is, correspondingly, 9, 73/7 and 37/3, we see that the maximum is 37/3.

20. Let f be defined by $f(0)=1$ and $f(x+\frac{1}{2})=2f(x)-\frac{1}{2}$. What is $f(-1)$?

(a) $\frac{3}{4}$ **(b)** $\frac{5}{8}$ **(c)** 2 **(d)** Cannot be determined from the given information
(e) none of the above

The answer is: **(b)**

Solution: $f(0)=f(-\frac{1}{2}+\frac{1}{2})$. So $2f(-\frac{1}{2})-\frac{1}{2}=1$. Solving for $f(-\frac{1}{2})$ gives $f(-\frac{1}{2})=\frac{3}{4}$. Then $f(-\frac{1}{2})=f(-1+\frac{1}{2})$ and it follows that $2f(-1)-\frac{1}{2}=\frac{3}{4}$. Now, solving for $f(-1)$ gives $f(-1)=\frac{5}{8}$.

21. If z is a complex number satisfying $z^5=1$, but $z^4\neq 1$, what is $z^4+z^3+z^2+z$?

(a) -1 **(b)** 2 **(c)** 0 **(d)** i **(e)** none of the above

The answer is: **(a)**

Solution: z is a root of the polynomial x^5-1. It's clear that 1 is a root of x^5-1, so $x-1$ is one factor. Using long division of polynomials we can see that x^5-1 factors as

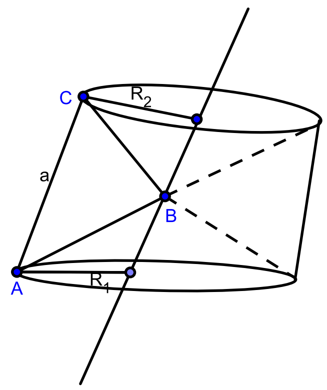
$(x-1)(x^4+x^3+x^2+x+1)$. Since $z^4 \neq 1$, $z \neq 1$, so z is a root of $x^4+x^3+x^2+x+1$. So $z^4+z^3+z^2+z+1=0$, so $z^4+z^3+z^2+z=-1$.

22. An equilateral triangle $\triangle ABC$ with a side a is revolved about a straight line l through B that doesn't intersect the triangle anywhere else. The angle α betweenAB and l is between $\frac{\pi}{4}$ and $\frac{\pi}{2}$. Find the surface area of the solid obtained by rotating $\triangle ABC$ about l.

(a) $\sqrt{3}\pi a^2 \sin\alpha$ **(b)** $2\sqrt{3}\pi a^2 \sin(30^\circ-\alpha)$ **(c)** $2\sqrt{3}\pi a^2 \cos(60^\circ-\alpha)$
(d) $\sqrt{3}\pi a^2 \cos\alpha$ **(e)** none of the above

The answer is: **(c)**

Solution



$S_1 = \pi a R_1$ is the surface area of the cone obtained by rotating AB about l where R_1 is the radius of the base.

$S_2 = \pi(R_1+R_2)a$ is the surface area of the portion of the cone obtained by rotating AC about l, where R_2 is the radius of the circle the point C traces out rotating about l.

$S_3 = \pi a R_2$ is the surface area of the cone obtained by rotating BC about l.

The surface area of the solid obtained is $S = S_1+S_2+S_3 = \pi a R_1 + \pi(R_1+R_2)a + \pi a R_2$. Hence

$$\begin{aligned}
S &= 2\pi a(R_1 + R_2) \\
&= 2\pi a(a\sin\alpha + a\sin(120^\circ - \alpha)) \\
&= 2\pi a^2(\sin\alpha + \sin(120^\circ - \alpha)) \\
&= 2\pi a^2 2\sin\frac{\alpha + (120^\circ - \alpha)}{2}\cos\frac{\alpha - (120^\circ - \alpha)}{2} \\
&= 4\pi a^2 \sin 60^\circ \cos(\alpha - 60^\circ) \\
&= 2\pi a^2\sqrt{3}\cos(\alpha - 60^\circ)
\end{aligned}$$

23. Find the smallest positive integer n such that every digit of $45n$ is 0 or 4.
(a) $n = 23456789$ **(b)** $n = 123456789$ **(c)** $n = 987654321$ **(d)** $n = 98765432$
(e) none of the above

The answer is: **(d)**

Solution: A number is divisible by 9 if the sum of the digits is divisible by 9. As 4 and 9 are relatively prime, if a nonzero natural number whose digits are 4's and 0's is divisible by 9, it must be have $9k$ 4's for some $k \geq 1$. So the smallest such number must be $444,444,444$. This number however is not divisible by 5. By adding a zero at the end, we obtain $4,444,444,440$ which is divisible by both 9 and 5. Hence, $n = \frac{4{,}444{,}444{,}440}{45} = 98765432$ is the smallest such number such that $45n$ consists only of digits 4 and 0.

24. Thirty bored students take turns walking down a hall that contains a row of closed lockers, numbered 1 to 30. The first student opens all the lockers; the second student closes all the lockers numbered 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30; the third student operates on the lockers numbered 3, 6, 9, 12, 15, 18, 21, 24, 27, 30: if a locker was closed, he opens it, and if a locker was open, he closes it; and so on. For the i^{th} student, he works on the lockers numbered by multiples of i: if a locker was closed, he opens it, and if a locker was open, he closes it. What is the number of lockers that remain open after all the students finish their walks?
(a) 2 **(b)** 4 **(c)** 5 **(d)** 12 **(e)** none of the above

The answer is: **(c)**

Solution Answer: 5. Solution: Let $d(i)$ be the number of positive integer divisors of the positive integer number i. The i^{th} locker remains open exactly when $d(i)$ is an odd number. This is so because the k^{th} student operates on the i^{th} locker if and only if k divides i. Now, we use that if $i = p_1^{k_1} p_2^{k_2} \dots p_j^{k_j}$ is the prime number decomposition of i (i.e., if we write i as the product of powers of distinct prime numbers) we have $d(i) = (1 + k_1)(1 + k_2)(1 + k_j)$. Therefore $d(i)$ is odd if and only if each of the powers $k_1, k_2, \dots, k_j$ is even. This is equivalent to i being a perfect (exact) square. Since there are 5 perfect squares less than 30 the answer is 5.

25. Let n and d be fixed integers. How many integers m are such that $n^2 + md$ is an exact square?

(a) Infinitely many m **(b)** 255 **(c)** 625 **(d)** no such m exists
(e) none of the above

The answer is: **(a)**

Solution: $(n + kd)^2 = n^2 + 2nkd + k^2d^2 = n^2 + d(2nd + k^2d)$. Hence, for any integer k, $(n + kd)^2$ is an exact square and has the form $n^2 + md$ with $m = 2nk + k^2d$. As the set of integers is infinite, the set of $m = 2nk + k^2d$ is infinite.