SOLUTIONS OF 2011 MATH OLYMPICS LEVEL II

1. $\sum_{i=2}^{n} \log\left(\frac{i+1}{i-1}\right) = \log\left(\frac{2+1}{2-1}\right) + \log\left(\frac{3+1}{3-1}\right) + \log\left(\frac{4+1}{4-1}\right) + \ldots + \log\left(\frac{n+1}{n-1}\right) =?$

(a) $\log\left(\frac{n(n+1)}{2}\right)$ **(b)** $\log(n+1)$ **(c)** $\log(n+1) + \log n - \log 2 - 1$

(d) $\frac{\log n(n+1)}{\log 2}$ **(e)** none of the above

The answer is: **(a)**

Solution:

$$\sum_{i=2}^{n} \log\left(\frac{i+1}{i-1}\right) = \sum_{i=2}^{n} \log(i+1) - \log(i-1)$$
$$= (\log 3 - \log 1) + (\log 4 - \log 2) + (\log 5 - \log 3) + \ldots + (\log(n-2) - \log(n-4)) +$$
$$(\log(n-1) - \log(n-3)) + (\log(n) - \log(n-2)) + (\log(n+1) - \log(n-1))$$
$$= -\log 1 - \log 2 + \log(n) + \log(n+1)$$
$$= \log(n+1) + \log(n) - \log 2$$
$$= \log\left(\frac{n(n+1)}{2}\right)$$

Note that the central negative terms cancelled out with one term on the left and another on the right, except the edges where one of them is missing. This is an example of the double telescopic sum. The very last equality uses logarithmic function properties.

2. Find an integer m in terms of n such that $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{1}{m}$.

(a) $m = 2n(n+1)(n+2)$ **(b)** $m = (n-1)n(n+2)$ **(c)** $m = 2(n+1)(n+2)$

(d) $m = n(n+1)(n+2)$ **(e)** none of the above

The answer is: **(c)**

Solution 1: We can find numbers A, B, and C such that

$$\frac{1}{k(k+1)(k+2)} = \frac{A}{k} + \frac{B}{k+1} + \frac{C}{k+2},$$

in this case $A = C = 1/2$, $B = -1$. Therefore,

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}\right)$$

By expanding the sum as in problem 1., the central negative terms cancelled out one term on the left and another on the right, except the edges where one of them is missing. This is another example of a double telescopic sum as in problem 1. Cancelling everything there is to be cancelled, we are left with,

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left[1 - \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}\right] = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right] = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

We conclude that $m = 2(n+1)(n+2)$.

Solution 2: A clever solution is to use telescopic sum of the simplest type, i.e.,

$$\sum_{k=1}^{n}(a_k - a_{k+1}) = (a_1 - a_2) + (a_2 - a_3) + \ldots + (a_{n-1} - a_n) + (a_n - a_{n+1}) = a_1 - a_{n+1}$$

Now if we use a partial fraction decomposition into two terms,

$$\frac{1}{k(k+1)(k+2)} = \frac{1/2}{k(k+1)} - \frac{1/2}{(k+1)(k+2)}.$$

Thus our sum S is a telescopic sum with term $a_k = \frac{1}{2k(k+1)}$,

$$S = \sum_{k=1}^{n}\left(\frac{1}{2k(k+1)} - \frac{1}{2(k+1)(k+2)}\right) = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

So $m = 2(n+1)(n+2)$.

3. Let $p(x) = 12x + 115$. Find the smallest integer $n \geq 1$ so that $p(n)$ is not a prime number.

(a) 5 **(b)** 3 **(c)** 10 **(d)** 11 **(e)** none of the above

The answer is: **(a)**

Solution: Note that $L(1)$, $L(2)$, $L(3)$, and $L(4)$ are all prime numbers but $L(5) = 175$ is not a prime number. So $x = 5$.

4. If $\sin(\alpha+\beta)=1$ and $\sin(\alpha-\beta)=\frac{1}{2}$, for $0<\alpha,\beta<\frac{\pi}{2}$, what is $\tan(\alpha+2\beta)$?

(a) $-\sqrt{3}$ **(b)** undefined **(c)** $\frac{1}{\sqrt{3}}$ **(d)** $\sqrt{3}$ **(e)** none of the above

The answer is: **(a)**

Solution: If $\sin(\alpha+\beta)=1$ with $0<\alpha,\beta<\frac{\pi}{2}$, then $\alpha+\beta=\frac{\pi}{2}$. Now if $\sin(\alpha-\beta)=\frac{1}{2}$ with $0<\alpha,\beta<\frac{\pi}{2}$, then $\alpha-\beta$ must be in quadrant I and therefore $\alpha-\beta=\frac{\pi}{6}$. Solving the system

$$\alpha+\beta=\frac{\pi}{2}$$

$$\alpha-\beta=\frac{\pi}{6}$$

gives $\alpha=\frac{\pi}{3}$ and $\beta=\frac{\pi}{6}$. So $\alpha+2\beta=\frac{2\pi}{3}$. Hence $\tan(\alpha+2\beta)=\tan\frac{2\pi}{3}=-\sqrt{3}$.

5. Let a and b be real numbers satisfying the equations

$$a+b=1, \qquad (a^2+b^2)(a^3+b^3)=26.$$

What is a^2+b^2?

(a) 26 **(b)** $\frac{4}{3}$ **(c)** $\frac{13}{3}$ **(d)** $\frac{5}{2}$ **(e)** none of the above

The answer is: **(c)**

Solution: We use the fact that $a^2+b^2=(a+b)^2-2ab=1-2ab$. Next we determine ab. Let $p=ab$. From $26=(a^2+b^2)(a^3+b^3)=((a+b)^2-2ab)((a+b)^3-3ab(a+b))=(1-2p)(1-3p)=6p^2-5p+1$. Thus $6p^2-5p-25=0$ i.e., $(3p+5)(2p-5)=0$ and so $p=-5/3$ or $p=5/2$. The solution $p=5/2$ gives $a^2+b^2=-4$, which is not possible. Hence $a^2+b^2=13/3$ with $p=-5/3$.

6. For how many integers n in the set $\{1,2,\dots,2011\}$ is n^4-n^3 a cube?

(a) 25 **(b)** 13 **(c)** 15 **(d)** 30 **(e)** none of the above

The answer is: **(b)**

Solution: As $n^4-n^3=n^3(n-1)$, $n-1$ has to be a cube, so the solution is an integer k such that $n=k^3+1\le 2011$. So $k^3\le 2010$. Since $12^3<2010$ and $13^3>2010$ it follows that there are thirteen k's, $k=0,1,2,\dots,12$, hence, 13 n's as required.

7. Joey has a slushy stand. He has found that if he charges \$1.25 per slushy he can sell 150 slushies per day. For each 2 cents raise in price per slushy, he sells 1 less slushy per day. It costs him 25 cents to make a slushy. What price will maximize his profit?

(a) \$1.00 **(b)** \$6.25 **(c)** \$1.06 **(d)** \$6.38 **(e)** none of the above

The answer is: **(e)**

Solution: Let n be the number of slushies sold per day and p the price of each slushy. Then the number of slushies is a linear function of the price through points $(1.25, 150)$ and slope $\frac{-1}{.02} = -50$ and we have the equation $n = -50p + 212.5$. The total cost of making n slushies is $.25n = .25(-50p + 212.5) = -12.5p + 53.125$. The total revenue is $pn = -50p^2 + 212.5p$. So now the profit as a function of p is

$$-50p^2 + 212.5p - (-12.5p + 53.125) = -50p^2 + 225p - 53.125,$$

is an open downward parabola with the maximum at the vertex point. Thus $p = -225/2 \cdot -50 = 2.25$. Hence a unit price of \$ 2.25 will maximize the profit.

8. An anagram of a word is another word (not necessarily belonging to the English language) made up of the same letters. So the word "mom" has three anagrams, namely "mmo", "mom", and "omm". How many anagrams are there of "Mississippi"?

(a) 11! **(b)** $\frac{11!}{16}$ **(c)** $\frac{11!}{1152}$ **(d)** $\frac{11!}{24}$ **(e)** none of the above

The answer is: **(c)**

Solution: We have a total of 11 words, so if we permute them, we get a total of 11! outcomes. But there are repetitions of "i" 4 times, "s" 4 times and "p" 2 times, so we must divide the total of possibilities by $4! \cdot 4! \cdot 2!$. Hence the number of possible anagrams is $\frac{11!}{4! \cdot 4! \cdot 2!} = \frac{11!}{1152}$.

9. If θ is an angle in the first quadrant, and $3\cos\theta - 4\sin\theta = 2$, what is the value of $3\sin\theta + 4\cos\theta$?

(a) 2 **(b)** $\sqrt{21}$ **(c)** $\sqrt{5}$ **(d)** -2 **(e)** none of the above

The answer is: **(b)**

Solution: It is easy to see that

$$(3\cos\theta + 4\cos\theta)^2 + (3\cos\theta - 4\sin\theta)^2 = 25.$$

Therefore $(3\cos\theta + 4\cos\theta)^2 = 21$. Hence $(3\cos\theta + 4\cos\theta) = \sqrt{21}$.

10. Which of the following is a solution to the equation $8\cos^2\theta - 8\cos\theta + 1 = 0$?

(a) $\frac{\pi}{4}$ **(b)** $\cos^{-1}\left(\frac{1}{2\sqrt{2}}\right)$ **(c)** $\frac{\pi}{3}$ **(d)** $\frac{\pi}{8}$ **(e)** none of the above

The answer is: **(e)**

Solution: Setting $x = \cos\theta$, we have $8x^2 - 8x + 1 = 0$. Using the quadratic formula, we have $x = \frac{8\pm\sqrt{32}}{16} = \frac{2\pm\sqrt{2}}{4}$. So $\cos\theta = \frac{2\pm\sqrt{2}}{4}$. Now $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos\frac{\pi}{8} = \sqrt{\frac{1+cos\frac{\pi}{4}}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}$, and $\cos\frac{\pi}{3} = \frac{1}{2}$. Hence none of the angles proposed works.

11. An integer-valued point in the xy-plane is a point (a, b) where both a and b are integers. How many integer-valued points are on or inside a circle of radius 4 centered at the origin?

(a) 50 **(b)** 21 **(c)** 49 **(d)** 7 **(e)** none of the above

The answer is: **(c)**

Solution: We want to count the number of points (a, b) where a and b are integers satisfying

$$\sqrt{a^2 + b^2} \leq 4.$$

The various possibilities are exhaustive: if $b = 0$ then a can be any integer from -4 to 4 so this is 9 points. If $b = \pm 1$ there 7 possible values for a, ranging from -3 through 3. When $b = \pm 2$ there are still 7 values for a, from -3 through 3. When $b = \pm 3$ there are now 5 values of a, ranging from -2 to 2. And finally the only possibility for $b = \pm 4$ is when $a = 0$. Adding all of this up gives

$$9 + 7 + 7 + 7 + 7 + 5 + 5 + 1 + 1 = 49.$$

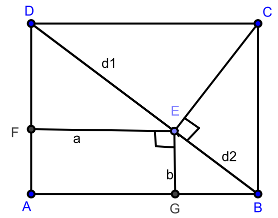
12. Suppose E is the foot of the perpendicular from C to diagonal BD in rectangle $ABCD$. If the lengths of perpendiculars from E to AD and AB are a and b, respectively, express the length d of diagonal BD in terms of a and b.

(a) $d = (a^{2/3} + b^{2/3})^{3/2}$ **(b)** $d = b/a$ **(c)** $d = a/b$

(d) $d = (a^2 + b^2)^{2/3}$ **(e)** none of the above

The answer is: **(a)**

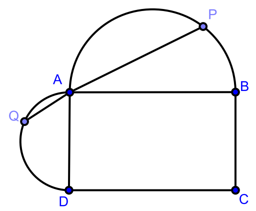
Solution: Let F and G be the feet of the perpendicular dropped from E onto sides DA and AB respectively. Let d_1 be the length of DE and d_2 be the length of EB.



Denote by θ the angle $\angle DBA$, note that
$\theta = \angle DBA = \angle DEF = \angle ECB = \angle EDC$.
Also note that, $d_1 = a\sec\theta$, and $d_2 = b\csc\theta$.
From $\triangle CDE$, we get $CE = d_1\tan\theta$.
From $\triangle BCE$, we get $CE = d_2\cot\theta$.
Therefore, $d_1 tan\theta = d_2\cot\theta$, i.e., $a\sec\theta\tan\theta = b\csc\theta\cot\theta$, finally $\tan^3\theta = b/a$.
Now $d = d_1 + d_2 = a\sec\theta + b\csc\theta = a\sqrt{1+\tan^2\theta} + b\sqrt{1+\cot^2\theta}$. Substituting $\tan^3\theta = b/a$ and $\cot^3\theta = a/b$ in the expression of d yields $d = (a^{2/3} + b^{2/3})^{3/2}$.

13. Semicircles are drawn on two sides of a rectangle $ABCD$ in which the longer side AB is twice the length of the shorter side AD, as shown. QAP is a line segment with segment QA of length $5cm$ and segment AP with length $24cm$. What is the length of the shorter side?
(a) $\frac{5}{\sqrt{2}}$ **(b)** 13 **(c)** 12 **(d)** $\frac{2}{\sqrt{2}}$ **(e)** none of the above

The answer is: **(b)**
Solution: This solution use similar triangles and the Pythagorean theorem.



First, we note that $\triangle DAQ$ is similar to $\triangle ABP$, symbolically, $\triangle DAQ \sim \triangle ABP$. Indeed, both triangles have one angle equal to $90°$ while at the vertex A we have $\angle QAD + \angle PAQ = 90°$ since $\angle DAB = 90°$. Since the sum of all angles in a triangle is $180°$ it follows that $\angle QAD = \angle PBA$.

Now since $\triangle DAQ \sim \triangle ABP$, we have

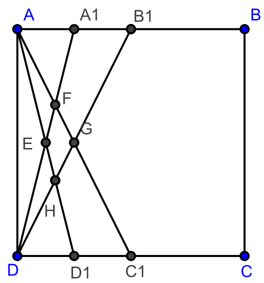
$$\frac{DA}{AB} = \frac{AQ}{BP} = \frac{DQ}{AP}.$$

Let $DA = x$. Since we are given that the side AB of the rectangle is twice longer than the side DA, we have $AB = 2DA = 2x$. The above identity can be written as

$$\frac{x}{2x} = \frac{5}{BP} = \frac{DQ}{24},$$

which shows that $DQ = 12$. Now, the Pythagorean theorem applied to $\triangle DAQ$ gives $x^2 = 12^2 + 5^2$, i.e., $x = 13$.

14. If $ABCD$ is a square with side AB of length $1cm$ and the segments AA_1,A_1B_1, C_1D_1, and D_1D all have length $\frac{1}{4}cm$, what is the area of the quadrilateral $EFGH$?
See the figure for the location of the points.



(a) $\frac{1}{12}$ **(b)** $\frac{1}{48}$ **(c)** $\frac{1}{96}$ **(d)** $\frac{1}{8}$ **(e)** none of the above

The answer is: **(b)**
Solution: Using similar triangles $\triangle AFA_1 \sim \triangle C_1DF$, the first one is twice smaller that the second, we see that F is at distance $2/3$ from DC. On the other hand, both, E and G are at distance $1/2$ from DC. Now using the areas of the triangles $\triangle DC_1F$, $\triangle DD_1E$, $\triangle DD_1H$ we see that the area of the quadrilateral $EFGH$ is

$$\text{Area} \triangle\, DC_1F - \text{Area} \triangle\, DD_1E - \text{Area} \triangle\, DC_1G + \text{Area} \triangle\, DD_1H$$

$$= \tfrac{1}{2}(\tfrac{2}{3}\cdot\tfrac{1}{2} - \tfrac{1}{2}\cdot\tfrac{1}{4} - \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{3}\cdot\tfrac{1}{4}) = \tfrac{1}{48}.$$

15. A writes a letter to B and does not receive an answer. Assuming that in general one letter in n is lost in the mail, where n is some fixed integer. Find the probability that B received the letter. It is to be assumed that B would have answered the letter if he had received it.

(a) $\frac{n-1}{n}$ **(b)** $\frac{n-1}{n^2} + \frac{n-1}{n}$ **(c)** $\frac{n-1}{n^2}$ **(d)** $\frac{n-1}{2n-1}$ **(e)** none of the above

The answer is: **(d)**

Solution: We know A did not receive an answer from B. We want to find the chance that the reason for this is that B's response to A was lost.

A sent a letter successfully and B sent a letter that was lost has $\frac{n-1}{n} \cdot \frac{1}{n}$ chance of happening.

The only other explanation for A not getting a response is that A sent a letter that was lost. This has a $\frac{1}{n}$ chance of happening.

Now we know that A did not get a response from B. The chance that the cause is that B response was lost is

$$\frac{\frac{n-1}{n} \cdot \frac{1}{n}}{\frac{n-1}{n} \cdot \frac{1}{n} + \frac{1}{n}} = \frac{n-1}{n-1+n} = \frac{n-1}{2n-1}.$$

16. In the game show "Let's Make a Deal", a contestant is presented with 3 doors. There is a prize behind one of the doors, and the host of the show knows which one. When the contestant makes a choice of door, at least one of the other doors will not have a prize, and the host will open a door (one not chosen by the contestant) with no prize. The contestant is given the option to change his choice after the host shows the door without a prize. If the contestant switches doors, what is the probability that he gets the door with the prize?

(a) $\frac{1}{3}$ **(b)** $\frac{1}{6}$ **(c)** $\frac{2}{3}$ **(d)** $\frac{1}{2}$ **(e)** none of the above

The answer is: **(c)**

Solution: Contestant's first choice has a 1/3 chance of being a prize door. If this is the case, when he switches, he is assured of choosing a door with no prize.

Contestant's first choice has a 2/3 chance of being a "no prize" door. If this is the case, when he switches, he is assured of choosing the door with the prize, because the other "no prize" door has been eliminated by the host.

Hence if the contestant switches doors, the probability he gets the prize is $1/3(0)+2/3(1) = 2/3$.

17. Suppose $f(x) = ax + b$. If $f(f(f(x))) = 125x + 155$, what is $a + b$?

(a) 5 **(b)** 13 **(c)** 17 **(d)** 10 **(e)** none of the above

The answer is: **(d)**

Solution: $f(f(f(x))) = a(a(ax+b)+b)+b = a^3x+a^2b+ab+b$. So if $f(f(f(x))) = 125x+155$, then $a = 5$ and $(25 + 5 + 1)b = 155$, so $b = 5$. Hence $a + b = 10$.

18. Consider a right triangle $\perp \triangle ABC$ with a and b being the perpendicular legs and c the hypotenuse. If the triangle ABC is inscribed in a circle with radius r, find $a^2 + b^2 + c^2$.

(a) $4r^2$ **(b)** $8r^2$ **(c)** $16r^2$ **(d)** $32r^2$ **(e)** none of the above

The answer is: **(b)**

Solution: By Pythagorean Theorem, $a^2 + b^2 = c^2$. Any right triangle inscribed in a circle has the diameter as its hypotenuse. So $c = 2r$ and $a^2+b^2 = (2r)^2 = 4r^2$. Hence $a^2+b^2+c^2 = 4r^2 + 4r^2 = 8r^2$.

19. Which of the following equations have the same graph?

$$I. \quad y = x + 4 \qquad II. \quad (x-4)y = x^2 - 16 \qquad III. \quad y = \frac{x^2-16}{x-4}$$

(a) I and II only **(b)** I and III only **(c)** II and III only
(d) I, II, and III **(e)** none of the above

The answer is: **(e)**

Solution: Graph I is a line.

Graph II is the intersection of two lines: $(x-4)y = x^2 - 16$ implies that $(x-4)y - (x-4)(x+4) = 0$, i.e., $(x-4)(y-x-4) = 0$. Thus $x = 4$ or $y = x + 4$.

Graph III is a line with a hole, it is in fact not defined at $x = 4$.

20. If $f(2x) = \frac{4}{4+x}$, then $2f(x) =$?

(a) $\frac{8}{4+x}$ **(b)** $\frac{4}{2+x}$ **(c)** $\frac{4}{4+x}$ **(d)** $\frac{16}{8+x}$ **(e)** none of the above

The answer is: **(d)**

Solution: Note that if $f(2x) = \frac{4}{4+x}$, then $f(x) = f(2\cdot\frac{1}{2}x) = \frac{4}{4+\frac{1}{2}x} = \frac{8}{8+x}$. Thus $2f(x) = \frac{16}{8+x}$.

21. If $f(\frac{x}{1-x}) = \frac{1}{x}$ for all $x \neq 0, 1$, then $f(\tan^2\theta) =$?

(a) $\frac{1}{\tan^2\theta}$ **(b)** $\tan^2\theta$ **(c)** $\sec^2\theta$ **(d)** $\csc^2\theta$ **(e)** none of the above

The answer is: **(d)**

Solution 1: Let $f(\frac{x}{1-x}) = \frac{1}{x}$ for all $x \neq 0, 1$. Set $\frac{x}{1-x} = \tan^2\theta$. Then solving for x gives

$$x = \frac{\tan^2\theta}{1+\tan^2\theta} = \frac{\tan^2\theta}{\sec^2\theta} = \frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{\cos^2}{1} = \sin^2\theta.$$

So $f(\tan^2\theta) = \frac{1}{\sin^2\theta} = \csc^2\theta$.

Solution 2: Let $f(\frac{x}{1-x}) = \frac{1}{x}$ for all $x \neq 0, 1$. Set $a = \frac{x}{1-x}$. Solving for x gives $x = \frac{a}{1+a}$. So then $f(a) = \frac{1}{\frac{a}{1+a}} = \frac{1+a}{a}$. Thus,

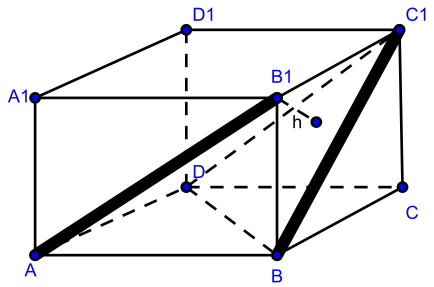
$$f(\tan^2\theta) = \frac{1+\tan^2\theta}{\tan^2\theta} = \frac{\sec^2\theta}{\tan^2\theta} = \csc^2\theta.$$

22. Find the distance between two nonintersecting diagonals on two adjacent sides of a cube with side a.

(a) 0 **(b)** a **(c)** $\frac{a}{\sqrt{3}}$ **(d)** $a\sqrt{2}$ **(e)** none of the above

The answer is: **(c)**

Solution 1:



Let AB_1 and BC_1 be the two nonintersecting adjacent diagonals (see figure). AB_1 is parallel to DC_1, so AB_1 is parallel to plane DBC_1. Let h be the orthogonal projection of B_1 onto plane DBC_1. Then h is the distance between the two diagonals AB_1 and BC_1. Note that h is the height of the pyramid $DBCB_1$ with volume that can be written in two ways

$$V = \frac{1}{3}h\cdot\mathcal{A}_{DBC} = \frac{1}{3}h\cdot\frac{\sqrt{3}}{2}a^2 \qquad (DB = BC_1 = DC_1 = a\sqrt{2})$$

$$V = \frac{1}{3}DC\cdot\mathcal{A}_{BB_1C_1} = \frac{1}{3}a\cdot\frac{1}{2}a^2$$

Thus $\frac{1}{3}h\cdot\frac{\sqrt{3}}{2}a^2 = \frac{1}{3}a\cdot\frac{1}{2}a^2$, i.e., $h = \frac{a}{\sqrt{3}}$.

Solution 2: One can consider the three dimensional coordinate space with origin $A(0,0,0)$ and axes AB, AD and AA_1. Since the distance between the two diagonals AB_1 and BC_1 is the distance from B_1 to plane BDC_1, it is enough to find and equation of plane BDC_1

using coordinates $B(a,0,0)$, $D(0,a,0)$, and $C_1(a,a,a)$. A point $M(x,y,z)$ is on the plane BDC_1 if and only if $\overrightarrow{BM} \cdot (\overrightarrow{BD} \times \overrightarrow{BC_1}) = 0$. Since $\overrightarrow{BD} \times \overrightarrow{BC_1} = (a^2, a^2, -a^2)$, we have $a^2(x-a) + a^2y - a^2z = 0$. Thus the equation of the plane BDC_1 is $x + y - z - a = 0$. Hence the distance h from $B_1 = (a,0,a)$ to plane BDC_1 is

$$h = \frac{|1 \cdot a + 1 \cdot 0 + (-1) \cdot a - a|}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{|a - a - a|}{\sqrt{3}} = \frac{a}{\sqrt{3}}.$$

Note: the distance d from a point $P(x_0, y_0, z_0)$ to a plane with equation $ax+by+cz+d = 0$ is given by the formula

$$d = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

23. Find all positive integers n such that $1 < n < 1000$ with the following property: the remainder when n is divided by 4 is 1, the remainder when n is divided by 7 is 1, the remainder when n is divided by 25 is 1.

(a) $n = 337$ **(b)** $n = 700$ **(c)** $n = 64$ **(d)** $n = 701$ **(e)** none of the above

The answer is: **(d)**

Solution: By hypothesis, n has remainder 1 when divided by 4, 7, and 25, that is, there exist positive integers p, q, and r such that

$$n = 4p + 1 = 7q + 1 = 25r + 1.$$

So then $n - 1$ has divisors 4, 7, and 25. Now since 4, 7, and 25 do not share any divisor, all three numbers are factors of $n - 1$, that is

$$n - 1 = 4 \cdot 7 \cdot 25 \cdot k = 700k, \text{ for some } k = 0, 1, 2 \ldots$$

Since we have $1 < n < 1000$, the only possible solutions are $k = 1$. So $n = 701$.

24. Observe that 4 can be expressed as the sum of natural numbers in 8 ways, taking into account the order of the terms:

$$4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1$$

How many such expressions are there for 2011?

(a) 2×2011 **(b)** 2^{2011} **(c)** 2^{2010} **(d)** 4020 **(e)** none of the above

The answer is: **(c)**

Solution 1. A very short solution of this problem is the following: Consider that n balls are arranged in line. Here are 4 balls ($\circ$):

$$\circ \circ \circ\circ$$

In each of the spaces between two balls, you may insert a divider. For example,

$$\circ \mid \circ\circ \mid \circ$$

represents 1+2+1. It is clear that you can obtain any decomposition of 4 this way, and given any decomposition, you get one of these representations. Now with n balls, there are $n-1$ spaces and for each of these spaces, you have the option of whether to insert a divider or not. So all together, you have 2^{n-1} variations.

Solution 2. One can also argue by induction. The case $n = 1$ gives only $1=2^0$ possibility, $n = 2$ gives $2 = 2^1$ possibilities, $n = 3$ gives $4 = 2^2$ possibilities, and the illustration of the problem 4 gives $8 = 2^3$ possibilities. We can then conjecture that $S(n) = 2^{n-1}$. For $n + 1$, if we write $n + 1 = (n - k) + k + 1$ for $k = 0, \ldots n + 1$, it is enough that we know for each k the number of decompositions of $n - k$ and adding them together will produce all the possibilities. But then by induction hypothesis we have that the possibilities for $n - k$ are 2^{n-k-1}. So

$$S(n + 1) = 1 + 1 + 2 + 2^2 + \ldots + 2^{n-2} + 2^{n-1}$$

Now knowing that the geometric sum $1 + 2 + 2^2 + \ldots + 2^n = 2^n - 1$, we have $S(n + 1) = 2^n$ as desired.

25. What are the last three digits of 5^{2011}?

(a) 125 **(b)** 525 **(c)** 625 **(d)** 025 **(e)** none of the above

The answer is: **(a)**

Solution: Let's look at the last three digits of the first few powers:

$$5, 25, 125, 625, 125, 625, 125, \ldots$$

Since other that the first two, the odd ones end in 125 and the even ones in 625, we claim that

$$5^{2n+1} \equiv 5^3 (\text{mod } 1000).$$

We prove our claim by induction. It is clear that the equation is satisfied for $n = 1$. Now assume that $5^{2k+1} \equiv 5^3 (\text{mod } 1000)$. Then $5^{2(k+1)+1} = 5^{2k+3} = 5^{2k+1}5^2 \equiv 5^3 \cdot 5^2 = 5^5 (\text{mod } 1000)$. Thus $5^{2(k+1)+1} \equiv 5^3 (\text{mod } 1000)$. Since we have an odd power, 2011, the answer is 125.