SOLUTIONS OF 2011 MATH OLYMPICS LEVEL I

1. The Easter Bunny creates a giant chocolate Easter egg that weighs 20 pounds. It is the exact same shape as a chocolate egg weighing 5 ounces. The egg that weighs 5 ounces is wrapped in 9 in^2 of foil. How much foil needs to be used to wrap the giant egg?

(a) 36 ft^2 **(b)** 16 ft^2 **(c)** 4 ft^2 **(d)** 1 ft^2

(e) none of the above

The answer is: **(d)**

Solution: If l is the linear scaling factor, then after converting pounds into ounces, we have $20 \cdot 16 = 5l^3$. So then $l^3 = \frac{20 \cdot 16}{5} = 4^3$ and $l = 4$.

Now to figure out how much foil x we need to wrap the giant egg, we use the linear scaling factor $l = 4$ and write $x = 9in^2 \cdot l^2 = 9in^2 \cdot 16 = 144in^2$. Now converting it in feet, 1 foot=12 inches, gives $x = \frac{144}{12 \cdot 12} = 1ft$.

2. Which of the following equations is the equation of a circle whose center is on the line that bisects the first quadrant and that has no intercepts?

(a) $x^2 - 6x + y^2 - 6y = -9$ **(b)** $x^2 + 6x + y^2 + 6y = -14$

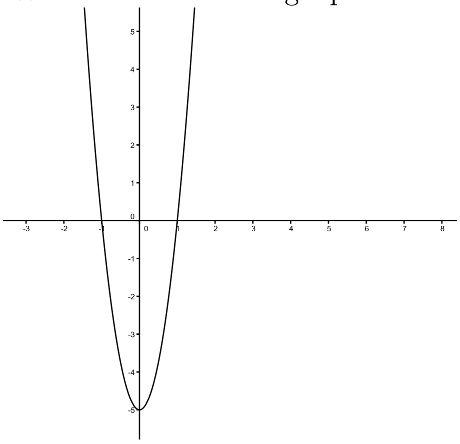
(c) $x^2 - 6x + y^2 - 4y = -12$ **(d)** $4x^2 - 24x + 4y^2 - 24y = -47$

(e) none of the above

The answer is: **(d)**

Solution: We proceed by elimination. It is clear that equation
(a) $x^2 - 6x + y^2 - 6y = -9$ has intercepts, in fact letting either $x = 0$ or $y = 0$ we get a solution, whereas (b), (c), and (d) have no intercepts. Now if the the center is on the line that bisects the first quadrant we should expect the center to be on the line $y = x$, i.e., to have coordinates of the form (x, x) and equation (c) doesn't have such center. Equations (b) and (d) satisfy both conditions. Now the center of (b), namely (-3,-3) is in the third quadrant, so it can't be (b). The center of (d) is (3,3) in the first quadrant, so (d) is the answer.

3. Which of the following equations could be the equation of the graph below.



(a) $y = -5x^2 - 5$ **(b)** $y = x^2 - 4x - 5$
(c) $y = x^4 + x^2 - 5$ **(d)** $y = -(x-1)(x+1)(x-5)$
(e) none of the above

The answer is: **(e)**
Solution: We again proceed by elimination. It is clear from the graph that the only zeros of the function are -1 and 1. We see that (a), (b), and (c) do not have zeros at both 1 and -1, and (d) has a zero at 5. So the answer is none of the above.

4. How many triples of real numbers (x, y, z) satisfy the equations

$$xy = z, \qquad yz = x, \qquad zx = y?$$

(a) 5 **(b)** 1 **(c)** 4 **(d)** infinitely many **(e)** none of the above

The answer is: **(a)**
Solution: If one of the number is zero, then clearly all of them must be zero. Suppose none of them is zero. Multiplying all equations we see that $x^2y^2z^2 = xyz$ and thus $xyz = 1$. Since $xy = z$ it follows that $z^2 = 1$, i.e., $z = \pm 1$. If $z = 1$ the other two of the given equations imply that either $x = y = 1$ or $x = y = -1$. If $z = -1$ the other two of the given equations imply that either $x = 1 = -y$ or $-x = 1 = y$. Thus the solution is given by the triples

$$\{(0,0,0),(1,1,1),(-1,-1,1),(1,-1,-1),(-1,1-1)\}.$$

So we have 5 such triples.

5. Let a and b be real numbers satisfying the equations

$$a + b = 1, \qquad a^3 + b^3 = 2011.$$

What is $a^2 + b^2$?

(a) 1 **(b)** 2010 **(c)** 1341 **(d)** 1851 **(e)** none of the above

The answer is: **(c)**

Solution: We use the fact that $a^2 + b^2 = (a+b)^2 - 2ab = 1 - 2ab$. Next we determine ab. Let $p = ab$. Then $2011 = a^3 + b^3 = (a+b)^3 - 3ab(a+b) = 1 - 3p$. So $p = -2010/3 = -670$. Thus $a^2 + b^2 = 1 - (2 \cdot -670) = 1341$.

6. Which of the following quadratic polynomials f satisfies the property: "$f(n)$ is an integer whenever n is an integer"?

(a) $f(x) = x^2 + \frac{x}{2} + \frac{1}{2}$ **(b)** $f(x) = \frac{x^2}{2} + \frac{x}{2} + \frac{1}{2}$ **(c)** $f(x) = \frac{x^2}{2} + \frac{x}{2}$
(d) $f(x) = x^2 - \frac{x}{2} - \frac{1}{2}$ **(e)** none of the above

The answer is: **(c)**

Solution: Note that the quadratic functions (a), (b) and (d) do not satisfy the desired condition at $x = 0$. Now for (c) $f(x) = \frac{x^2}{2} + \frac{x}{2} = \frac{x^2+x}{2} = \frac{x(x+1)}{2}$,
but $x(x+1)$ is even whenever x is an integer. Therefore $\frac{x(x+1)}{2}$ is integer whenever x is integer. Hence (c) is the required quadratic.

7. A drawer contains red socks and black socks. When two socks are drawn at random, the probability that both are red is $\frac{1}{2}$. How small can the number of socks in the drawer be if the number of black socks is even?

(a) 7 **(b)** 14 **(c)** 21 **(d)** 25 **(e)** none of the above

The answer is: **(c)**

Solution: Let $2b$ be the number of black socks and r the number of red socks. Then the probability to get only red socks is $\frac{r}{2b+r} \cdot \frac{r-1}{2b+r-1} = \frac{1}{2}$.

$$2r(r-1) = (2b+r)(2b+r-1)$$

$$2r^2 - 2r = r^2 + (4b-1)r + 4b^2 - 2b$$

$$r^2 - (1+4b)r - (4b^2 - 2b) = 0$$

Using the quadratic formula, we have

$$r = \frac{1+4b \pm \sqrt{(1+4b)^2 + 4(4b^2 - 2b)}}{2}$$

$$r = \frac{1+4b \pm \sqrt{32b^2+1}}{2}$$

Now for $b = 1, 2$, $\sqrt{32b^2+1}$ is not an integer. But $b = 3$ gives $\sqrt{32b^2+1} = 17$ and so $r = \frac{1+4\cdot3+17}{2} = \frac{30}{2} = 15$.

Black socks number is $2b = 6$ and red socks number is 15 for a total of 21 socks.

8. What is the smallest number of coins (pennies, nickels, dimes, quarters, and half dollars) with which you can pay out any amount from 1 cent to 99 cents?

(a) 8 **(b)** 9 **(c)** 10 **(d)** 12 **(e)** none of the above

The answer is: **(b)**

Solution: The easiest way to see the answer is to start with small amounts and work our way up. So the only way to pay out 4 cents is with 4 pennies so we need **at least** 4 pennies. To have more than 4 pennies, however, would be wasteful as we can start to use nickels after this. With 4 pennies and one nickel we can make every amount less than a dime. Once we reach 10 cents there is no need to use more nickels and pennies so we start to use dimes. With 4 pennies, 1 nickel, and 1 dime we get every amount up to 19 cents. With 4 pennies, 1, nickel and 2 dimes we get up to 29 cents. At this point we introduce 1 quarter and then finally 1 half dollar to get every amount up to and including 99 cents. So we need a total of 9 coins. Note that instead of two dimes one could use 2 nickels.

9. A fair coin is tossed 10 times. What is the probability that exactly five of the tosses come up heads and five of them tails?

(a) $\frac{1}{2}$ **(b)** $\frac{63}{256}$ **(c)** $\frac{64}{256}$ **(d)** $\frac{5}{128}$ **(e)** none of the above

The answer is: **(b)**

Solution: The total number of possible outcomes is 2^{10} since each tosses must be either heads or tails, and each outcome is equally likely. The total number of ways in which five of them can be heads is the binomial coefficient

$$\binom{10}{5}.$$

We have

$$\binom{10}{5} = \frac{10!}{5!5!}$$

Thus the answer is

$$\frac{10!}{5!5!2^{10}} = \frac{252}{1024} = \frac{63}{256}.$$

10. How many zeros are at the end of 17!? (Recall that $n!$ is the product of the first n positive integers, that is $2! = 2 \cdot 1$, $3! = 3 \cdot 2 \cdot 1$, and so on)

(a) 2 **(b)** 3 **(c)** 17 **(d)** 21 **(e)** none of the above

The answer is: **(b)**

Solution: Each zero at the end of $n!$ represents one time that 10 divides $n!$. So the question to answer is how many times does 10 divide 17!? Since $10 = 5 \cdot 2$ we need to count how many times 5 divides 17! and then how many times 2 divides 17!. Looking at the number of factors in 17! that are divisible by 5 we find 3, namely 5, 10, and 15. There are certainly more than 3 factors of 2 in 17! as the factor 16 alone has 4 two's. So there will be exactly 3 zeroes at the end of 17!.

11. A train moving 45 miles per hour meets and is passed by a train moving 35 miles per hour in a opposite direction. A passenger in the first train sees the second train takes 6 seconds to pass him. How long is the second train?

(a) 701 feet **(b)** 698 feet **(c)** 704 feet **(d)** 710 feet

(e) none of the above

The answer is: **(c)**

Solution: As the two trains are moving at the same time the speed of the other train should appear to the passenger as 45 mph +35 mph. Now using convertion of 1 mile = 5280 feet and 1 hour=3600 seconds, we have

$$\frac{(45 + 35) \cdot 5280 \cdot 6}{3600} = 704 \text{ feet.}$$

12. Each morning Boris walks to school at a constant rate. At one-fourth of the way he passes the tractor station. At one-third of the way, the railroad station. At the tractor station its clock shows 7:30 A.M., and at the railroad station its clock shows 7:35 A.M. When does Boris reach school?

(a) 8:10 A.M. **(b)** 8:15 A.M. **(c)** 8:20 A.M. **(d)** 8:25 A.M.

(e) none of the above

The answer is: **(b)**

Solution: Boris's speed is $(\frac{1}{3} - \frac{1}{4}) \div 5 = \frac{1}{60}$ (total distance per minute). So he needs 60 minutes for the total distance, i.e., 15 minutes for the first one-fourth of the way and 40 minutes for the last two third of the way. So Boris reaches school at 7:35 A.M + 40 minutes=8:15 A.M.

13. If Peter Piper can pick a peck of pickled peppers in 2 hours, and Tom Thumb can do it in 3 hours, and Mary Mary can do it in 5 hours, how long will it take them working all together?

(a) 31/3 hours **(b)** 30/31 of an hour **(c)** 16/15 hours
(d) one hour and 2 minutes **(e)** none of the above

The answer is: **(b)**

Solution: If t is the time for all of them to complete the job together, we have

$$(1/2)t + (1/3)t + (1/5)t = 1$$

Solving for t gives $t = 30/31$ of an hour.

14. If r_1 and r_2 are solutions of the equation $x = a\sqrt{x + k^2}$, then $r_1 + r_2 =$?

(a) a^2 **(b)** $\frac{k^2}{1-a}$ **(c)** 0 **(d)** $a\sqrt{a^2 + 4ak^2}$ **(e)** none of the above

The answer is: **(a)**

Solution: From $x = a\sqrt{x + k^2}$, squaring both sides of the equality gives $x^2 = a^2(x + k^2)$, i.e.,

$$x^2 - a^2x - a^2k^2 = 0 \qquad (*).$$

Note that for every quadratic equation $ax^2 + bx + c = 0$, any two solutions x_1 and x_2 if exist, satisfy $x_1 + x_2 = \frac{-b}{a}$ and $x_1 \cdot x_2 = \frac{c}{a}$. So any two solutions r_1 and r_2 of the quadratic equation $(*)$ satisfy $r_1 + r_2 = a^2$.

15. For all x such that $x \neq 5, \frac{3}{2}, \frac{-3}{2}, 0$, what is $\frac{2}{x-5} - \frac{14}{2x^2-13x+15}$ equal to?

(a) $\frac{4}{2x-3}$ **(b)** $\frac{8x+12}{4x^2-9}$ **(c)** $\frac{\frac{4}{x}}{2-\frac{3}{x}}$
(d) all of the above **(e)** none of the above

The answer is: **(d)**

Solution:

$$\begin{aligned}\frac{2}{x-5}-\frac{14}{2x^2-13x+15} &= \frac{2}{x-5}-\frac{14}{(x-5)(2x-3)}\\ &= \frac{2(2x-3)-14}{(x-5)(2x-3)}\\ &= \frac{4x-20}{(x-5)(2x-3)}\end{aligned}$$

Now when $x \neq 5, \pm\frac{3}{2}, 0$, we do have

$$\frac{2}{x-5}-\frac{14}{2x^2-13x+15} = \frac{4x-20}{(x-5)(2x-3)} = \frac{4(x-5)}{(x-5)(2x-3)} = \frac{4}{2x-3}$$

So it does satisfy (a). The two lines below show that it also does satisfy (b) and (c).

$$\frac{8x+12}{4x^2-9} = \frac{4(2x+3)}{(2x+3)(2x-3)} = \frac{4}{2x-3}$$

$$\frac{\frac{4}{x}}{2-\frac{3}{x}} = \frac{\frac{4}{x}\cdot x}{(2-\frac{3}{x})\cdot x} = \frac{4}{2x-\frac{3x}{x}} = \frac{4}{2x-3}$$

Hence the correct answer is all of the above.

16. Let x, y, z be real numbers. Suppose that $(x+1)(y+1)(z+1) \neq 0$, and

$$\frac{x}{x+1}+\frac{y}{y+1}+\frac{z}{z+1} = 1.$$

Find all possible values of the quantity $2xyz + xy + yz + zx$ for all x, y, z with the above properties.

(a) 1 **(b)** 0, 2, or 3 **(c)** 0 **(d)** infinitely many values
(e) none of the above

The answer is: **(a)**

Solution: Multiply both side of the equation by $(x+1)(y+1)(z+1)$, the non-zero common denominator, and simplify,

$$\begin{aligned}x(y+1)(z+1)+y(x+1)(z+1)+z(x+1)(y+1) &= (x+1)(y+1)(z+1)\\ 3xyz+2xy+2yz+2zx+x+y+z &= xyz+xy+yz+zx+x+y+z+1\end{aligned}$$

Now collect and cancel similar terms, to obtain

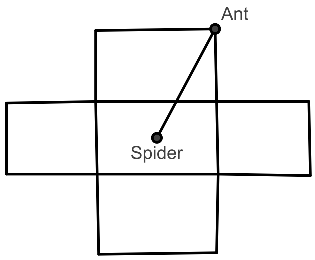
$$2xyz+xy+yz+zx = 1.$$

17. Suppose a glass has a square base of side 2 units and height 2 units. A spider is standing at the center of the bottom of a glass. The spider wants to reach a delicious ant that is standing at one of the top corners of the glass. Assume the spider walks at a constant speed on the surface of the glass and the ant, unaware of the danger, does not move. What distance should the spider walk to have her meal as quickly as possible?

(a) $2+\sqrt{2}$ **(b)** $1+\sqrt{5}$ **(c)** $\sqrt{10}$ **(d)** $2\sqrt{2}$ **(e)** none of the above

The answer is: **(c)**

Solution: To answer this question more quickly, flatten the glass, and join the center of the square base to a "corner" on the rim of the glass with a straight line (see figure). That is the shortest path. Its length, by Pythagorean theorem since it is the hypotenuse of a right triangle with sides 1 and 3 units long, is $\sqrt{10}$.



18. A quartet with 4 different instruments wants to play something (silence doesn't count) with each possible combination of instruments. How many ways can this be done?

(a) 15 **(b)** 10 **(c)** 24 **(d)** 18 **(e)** none of the above

The answer is: **(a)**

Solution: Since the silence doesn't count, the quartet might use exactly one instrument and in that case the possibilities are

$$\binom{4}{1} = \frac{4!}{1!3!} = 4,$$

or exactly two instruments with possibilities

$$\binom{4}{2} = \frac{4!}{2!2!} = 6,$$

or exactly three instruments with possibilities

$$\binom{4}{3} = \frac{4!}{3!1!} = 4,$$

or finally exactly four instruments with possibility

$$\binom{4}{4} = \frac{4!}{4!0!} = 1.$$

Thus the total possibilities are $4 + 6 + 4 + 1 = 15$.

19. Which x is a solution of the equation $x^3 + 3x - 4 = 0$?

(a) $x = (2 + \sqrt{5})^{1/3} + (2 - \sqrt{5})^{1/3}$ **(b)** $x = (2 + \sqrt{5})^{1/3} - (2 - \sqrt{5})^{1/3}$

(c) 0 **(d)** 2 **(e)** none of the above

The answer is: **(a)**

Solution: We proceed by testing out each given x. It is clear that cases (c) and (d) are eliminated. Let see if (a) satisfies the equation:

$$\begin{aligned}
x^3 &= \left[(2+\sqrt{5})^{1/3} + (2-\sqrt{5})^{1/3}\right]^3 \\
&= 2+\sqrt{5} + 3(2+\sqrt{5})^{2/3}(2-\sqrt{5})^{1/3} + 3(2+\sqrt{5})^{1/3}(2-\sqrt{5})^{2/3} + (2-\sqrt{5}) \\
&= 4 + 3(2+\sqrt{5})^{1/3}(2-\sqrt{5})^{1/3}[(2+\sqrt{5})^{1/3} + (2-\sqrt{5})^{1/3}] \\
&= 4 + 3\left[(2+\sqrt{5})(2-\sqrt{5})\right]^{1/3} x \\
&= 4 + 3(-1)^{1/3}x = 4 - 3x
\end{aligned}$$

So $x = (2 + \sqrt{5})^{1/3} + (2 - \sqrt{5})^{1/3}$ satisfies $x^3 + 3x - 4 = 0$.

Note: it is clear that $x = 1$ is a solution of $x^3 + 3x - 4 = 0$. Furthermore $x^3 + 3x - 4 = (x - 1)(x^2 + x + 4)$, so the other two solutions are roots of the quadratic $x^2 + x + 4 = 0$, which does not have any real roots. Therefore we can conclude that $(2 + \sqrt{5})^{1/3} + (2 - \sqrt{5})^{1/3} = 1$ and that (b) does not satisfy the equation.

20. Find real numbers A, B and C so that for all real numbers $x \neq 0, 3, -1$, the following identity holds

$$\frac{1}{x(x-3)(x+1)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+1}$$

(a) $A = -1/3, B = 1/12, C = 1/14$ **(b)** $A = 1/3, B = -1/12, C = 1/14$

(c) $A = -1/2, B = 1/4, C = 1/12$ **(d)** $A = -1/3, B = 1/12, C = 1/4$

(e) none of the above

The answer is: **(d)**

Solution: This is a standard problem on partial fraction decomposition. We multiply the right hand side and the left hand side by the common denominator $x(x-3)(x+1)$, and we are left with the identity

$$1 = A(x-3)(x+1) + Bx(x+1) + Cx(x-3).$$

Next we use the fact that two polynomials are equal if and only if all their coefficients coincide. After simplifying and collecting terms on the right-hand-side, we get

$$0x^2 + 0x + 1 = (A+B+C)x^2 + (-2A+B-3C)x - 3A.$$

This translates into a system of three linear equations in the 3 unknowns A, B, and C, namely:

$$\begin{aligned} A+B+C &= 0, \\ -2A+B-3C &= 0, \\ -3A &= 1. \end{aligned}$$

We deduce immediately that $A = -1/3$ and substituting it in the two first equation gives a system of two linear equations in 2 unknowns B and C

$$\begin{aligned} B+C &= 1/3, \\ B-3C &= -2/3. \end{aligned}$$

which can be solved by elimination (or any of your favorite methods) and we get $B = 1/12$ and $C = 1/4$.

21. An equation of the parabola through the three points $(0,1)$, $(1,4)$, $(2,9)$ is:

(a) $f(x) = -x^2+4x+1$ **(b)** $f(x) = x^2-2x+1$ **(c)** $f(x) = 2x^2+2x+1$

(d) $f(x) = x^2+2x+1$ **(e)** none of the above

The answer is: **(d)**

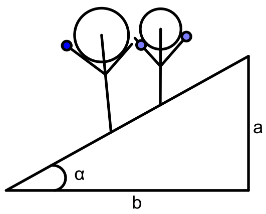
Solution 1: This can be done just by checking that only (d) $f(x) = x^2+2x+1$ passes through the 3 given points. In fact, $f(0) = 1$, $f(1) = 4$ and $f(2) = 9$.

Solution 2: Let $f(x) = ax^2+bx+c$ be the equation of the parabola. We are given $f(0) = 1$ (i.e., $c = 1$), $f(1) = 4$ and $f(2) = 9$. The two last equations give the system:

$$\begin{aligned} a+b+1 &= 4, \\ 4a+2b+1 &= 9. \end{aligned}$$

Solving it gives $a = 1$ and $b = 2$. Hence $f(x) = x^2+2x+1$.

22. The unit used for measuring the steepness of a hill is the "grade". A grade of a to b means the hill rises a vertical units for every b horizontal units. If at some point, the hill is $3ft$ above the horizontal and the angle of elevation α to that point is 30°, what is the approximate grade to the nearest integer of this hill?



(a) 3 to 5 **(b)** 3 to 6 **(c)** 3 to 7 **(d)** 3 to 8 **(e)** 7 to 10

The answer is: **(a)**

Solution: Having $a = 3$, it is enough to find the corresponding value of b. Since $\alpha = 30°$, the triangle is half of an equilateral triangle, so the hypotenuse must be 6. Using the Pythagorean Theorem tells us that $b = \sqrt{27}$. Since 5 is closer to $\sqrt{27}$ than 6, 7 or 8, and $\frac{3}{5}$ is closer to $\frac{3}{\sqrt{27}}$ than $\frac{7}{10}$, (a) is the closest approximation.

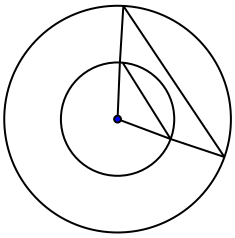
23. A circular corral was constructed with $600ft$ of barbed wire fence. Another circular corral with twice the area was constructed with the same type of fence. How much barbed wire was needed for the second?

(a) $1200ft$ **(b)** $600\sqrt{2}$ **(c)** 2400 **(d)** $1200\sqrt{2}$ **(e)** none of the above

The answer is: **(b)**

Solution: If R_1 is the radius of the corral C_1 enclosed with $600ft$ of fencing and R_2 the radius of the corral C_2 such that the area of C_2 is the double of C_1, then $\pi R_2^2 = 2\pi R_1^2$, i.e. $R_2 = R_1\sqrt{2}$. We know that the circumference of C_1 is $2\pi R_1 = 600$, so the circumference of C_2 is $2\pi R_2 = 2\pi R_1\sqrt{2} = 600\sqrt{2}$. Thus $600\sqrt{2}ft$ of barbed wire are needed.

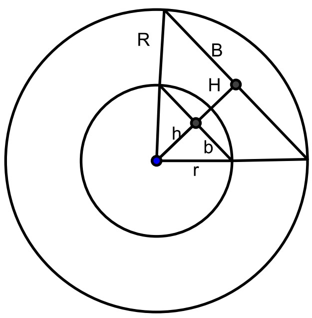
24. Two concentric circles are drawn, the smaller with radius r and the larger with radius R. A central angle is drawn, such that the rays intersect each circle as shown in the figure. Given that the area of the triangle in the larger circle is three times the area of the corresponding triangle in the smaller circle, give the area lying between the two circles in terms of r.



(a) $2\pi r^2$ **(b)** $\sqrt{3}\pi r^2$ **(c)** $(\sqrt{3}-1)\pi r^2$ **(d)** πr^2 **(e)** none of the above

The answer is: **(a)**

Solution 1:



Note that the small triangle (height h and base b) and the big one (height H and base B) are similar. So

$$\frac{H}{h} = \frac{B}{b} = \frac{R}{r}.$$

Since the area of the big one is three times the area of the small one, we have $\frac{1}{2}BH = 3\frac{1}{2}bh$, i.e., $\frac{H}{h}\frac{B}{b} = 3$. Thus, using the relation of similar triangles, we have $\frac{R^2}{r^2} = 3$, i.e., $R^2 = 3r^2$. Hence the area lying between the two circles is $\pi R^2 - \pi r^2 = 3\pi r^2 - \pi r^2 = 2\pi r^2$.

Solution 2: Using the area of the triangle in terms of the angle gives $\frac{1}{2}R^2 \sin\theta = 3\frac{1}{2}r^2 \sin\theta$. So $R = r\sqrt{3}$. Thus $\pi R^2 - \pi r^2 = \pi(r\sqrt{3})^2 - \pi r^2 = 2\pi r^2$.

25. A problem to celebrate the new year 2011, what are the last three digits of 2011^5?
(a) 121 **(b)** 051 **(c)** 131 **(d)** 641 **(e)** none of the above

The answer is: **(b)**

Solution: Thinking of 2011 as 2000+11, when we raise this to the fifth power, all terms in the expansion will have a factor of 2000 in them except the last 11^5. Any number with the factor of 2000 ends with 3 zeros and so does not affect the three last digits. As to 11^5 its last three digits are 051 and so 051 are the three last digits of 2011^5.