SAGINAW VALLEY STATE UNIVERSITY
2011 MATH OLYMPICS-LEVEL II

1. $\displaystyle\sum_{i=2}^{n} \log\left(\frac{i+1}{i-1}\right) = \log\left(\frac{2+1}{2-1}\right) + \log\left(\frac{3+1}{3-1}\right) + \log\left(\frac{4+1}{4-1}\right) + \ldots + \log\left(\frac{n+1}{n-1}\right) = ?$

(a) $\log\left(\frac{n(n+1)}{2}\right)$ **(b)** $\log(n+1)$ **(c)** $\log(n+1) + \log n - \log 2 - 1$
(d) $\frac{\log n(n+1)}{\log 2}$ **(e)** none of the above

2. Find an integer m in terms of n such that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{1}{m}$.

(a) $m = 2n(n+1)(n+2)$ **(b)** $m = (n-1)n(n+2)$ **(c)** $m = 2(n+1)(n+2)$
(d) $m = n(n+1)(n+2)$ **(e)** none of the above

3. Let $p(x) = 12x + 115$. Find the smallest integer $n \geq 1$ so that $p(n)$ is not a prime number.
(a) 5 **(b)** 3 **(c)** 10 **(d)** 11 **(e)** none of the above

4. If $\sin(\alpha + \beta) = 1$ and $\sin(\alpha - \beta) = \frac{1}{2}$, for $0 < \alpha, \beta < \frac{\pi}{2}$, what is $\tan(\alpha + 2\beta)$?
(a) $-\sqrt{3}$ **(b)** undefined **(c)** $\frac{1}{\sqrt{3}}$ **(d)** $\sqrt{3}$ **(e)** none of the above

5. Let a and b be real numbers satisfying the equations

$$a + b = 1, \qquad (a^2 + b^2)(a^3 + b^3) = 26.$$

What is $a^2 + b^2$?
(a) 26 **(b)** $\frac{4}{3}$ **(c)** $\frac{13}{3}$ **(d)** $\frac{5}{2}$ **(e)** none of the above

6. For how many integers n in the set $\{1, 2, \ldots, 2011\}$ is $n^4 - n^3$ a cube?
(a) 25 **(b)** 13 **(c)** 15 **(d)** 30 **(e)** none of the above

7. Joey has a slushy stand. He has found that if he charges \$1.25 per slushy he can sell 150 slushies per day. For each 2 cents raise in price per slushy, he sells 1 less slushy per day. It costs him 25 cents to make a slushy. What price will maximize his profit?

(a) \$1.00 **(b)** \$6.25 **(c)** \$1.06 **(d)** \$6.38 **(e)** none of the above

8. An anagram of a word is another word (not necessarily belonging to the English language) made up of the same letters. So the word "mom" has three anagrams, namely "mmo", "mom", and "omm". How many anagrams are there of "Mississippi"?

(a) 11! **(b)** $\frac{11!}{16}$ **(c)** $\frac{11!}{1152}$ **(d)** $\frac{11!}{24}$ **(e)** none of the above

9. If θ is an angle in the first quadrant, and $3\cos\theta - 4\sin\theta = 2$, what is the value of $3\sin\theta + 4\cos\theta$?

(a) 2 **(b)** $\sqrt{21}$ **(c)** $\sqrt{5}$ **(d)** -2 **(e)** none of the above

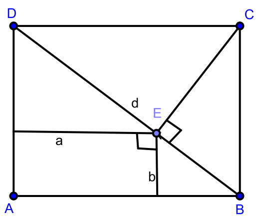
10. Which of the following is a solution to the equation $8\cos^2\theta - 8\cos\theta + 1 = 0$?

(a) $\frac{\pi}{4}$ **(b)** $\cos^{-1}\left(\frac{1}{2\sqrt{2}}\right)$ **(c)** $\frac{\pi}{3}$ **(d)** $\frac{\pi}{8}$ **(e)** none of the above

11. An integer-valued point in the xy-plane is a point (a, b) where both a and b are integers. How many integer-valued points are on or inside a circle of radius 4 centered at the origin?

(a) 50 **(b)** 21 **(c)** 49 **(d)** 7 **(e)** none of the above

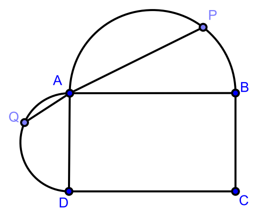
12. Suppose E is the foot of the perpendicular from C to diagonal BD in rectangle $ABCD$. If the lengths of perpendiculars from E to AD and AB are a and b, respectively, express the length d of diagonal BD in terms of a and b.



(a) $d = (a^{2/3} + b^{2/3})^{3/2}$ **(b)** $d = b/a$ **(c)** $d = a/b$

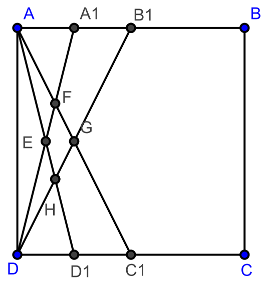
(d) $d = (a^2 + b^2)^{2/3}$ **(e)** none of the above

13. Semicircles are drawn on two sides of a rectangle $ABCD$ in which the longer side AB is twice the length of the shorter side AD, as shown. QAP is a line segment with segment QA of length $5cm$ and segment AP with length $24cm$. What is the length of the shorter side?



(a) $\frac{5}{\sqrt{2}}$ **(b)** 13 **(c)** 12 **(d)** $\frac{2}{\sqrt{2}}$ **(e)** none of the above

14. If $ABCD$ is a square with side AB of length $1cm$ and the segments AA_1,A_1B_1, C_1D_1, and D_1D all have length $\frac{1}{4}cm$, what is the area of the quadrilateral $EFGH$?
See the figure for the location of the points.



(a) $\frac{1}{12}$ **(b)** $\frac{1}{48}$ **(c)** $\frac{1}{96}$ **(d)** $\frac{1}{8}$ **(e)** none of the above

15. A writes a letter to B and does not receive an answer. Assuming that in general one letter in n is lost in the mail, where n is some fixed positive integer, find the probability that B received the letter. It is to be assumed that B would have answered the letter if he had received it.

(a) $\frac{n-1}{n}$ **(b)** $\frac{n-1}{n^2} + \frac{n-1}{n}$ **(c)** $\frac{n-1}{n^2}$ **(d)** $\frac{n-1}{2n-1}$ **(e)** none of the above

16. In the game show "Let's Make a Deal", a contestant is presented with 3 doors. There is a prize behind one of the doors, and the host of the show knows which one. When the contestant makes a choice of door, at least one of the other doors will not have a prize, and the host will open a door (one not chosen by the contestant) with no prize. The contestant is given the option to change his choice after the host shows the door without a prize. If the contestant switches doors, what is the probability that he gets the door with the prize?

(a) $\frac{1}{3}$ **(b)** $\frac{1}{6}$ **(c)** $\frac{2}{3}$ **(d)** $\frac{1}{2}$ **(e)** none of the above

17. Suppose $f(x) = ax + b$. If $f(f(f(x))) = 125x + 155$, what is $a + b$?

(a) 5 **(b)** 13 **(c)** 17 **(d)** 10 **(e)** none of the above

18. Consider a right triangle $\perp \triangle ABC$ with a and b being the perpendicular legs and c the hypotenuse. If the triangle ABC is inscribed in a circle with radius r, find $a^2 + b^2 + c^2$.

(a) $4r^2$ **(b)** $8r^2$ **(c)** $16r^2$ **(d)** $32r^2$ **(e)** none of the above

19. Which of the following equations have the same graph?

$$I. \quad y = x + 4 \qquad II. \quad (x-4)y = x^2 - 16 \qquad III. \quad y = \frac{x^2 - 16}{x - 4}$$

(a) I and II only **(b)** I and III only **(c)** II and III only
(d) I, II, and III **(e)** none of the above

20. If $f(2x) = \frac{4}{4+x}$, then $2f(x) = ?$

(a) $\frac{8}{4+x}$ **(b)** $\frac{4}{2+x}$ **(c)** $\frac{4}{4+x}$ **(d)** $\frac{16}{8+x}$ **(e)** none of the above

21. If $f(\frac{x}{1-x}) = \frac{1}{x}$ for all $x \neq 0, 1$, then $f(\tan^2\theta) = ?$

(a) $\frac{1}{\tan^2\theta}$ **(b)** $\tan^2\theta$ **(c)** $\sec^2\theta$ **(d)** $\csc^2\theta$ **(e)** none of the above

22. Find the distance between two nonintersecting diagonals on two adjacent sides of a cube with side a.

(a) 0 **(b)** a **(c)** $\frac{a}{\sqrt{3}}$ **(d)** $a\sqrt{2}$ **(e)** none of the above

23. Find all positive integers n such that $1 < n < 1000$ with the following property: the remainder when n is divided by 4 is 1, the remainder when n is divided by 7 is 1, the remainder when n is divided by 25 is 1.

(a) $n = 337$ **(b)** $n = 700$ **(c)** $n = 64$ **(d)** $n = 701$ **(e)** none of the above

24. Observe that 4 can be expressed as the sum of natural numbers in 8 ways, taking into account the order of the terms:

$$4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1$$

How many such expressions are there for 2011?

(a) 2×2011 **(b)** 2^{2011} **(c)** 2^{2010} **(d)** 4020 **(e)** none of the above

25. What are the last three digits of 5^{2011}?

(a) 125 **(b)** 525 **(c)** 625 **(d)** 025 **(e)** none of the above