Saginaw Valley State University

2010 Math Olympics – Level II Solutions

1. If f is a nonzero function (this means there is at least one x with $f(x) \neq 0$) of real numbers such that $f(x+y) = f(x)f(y)$, what are the possible values for $f(0)$?

(a) any real number is possible **(b)** any positive real number is possible

(c) $f(0)$ must be 0 **(d)** $f(0)$ must be 1

(e) $f(0)$ could be 0 or 1

SOLUTION **(d)**: $f(0+0) = f(0)f(0)$, so $f(0) = f(0)^2$ and $f(0)$ must be 0 or 1. If $f(0) = 0$, then for all x, we would have $f(x) = f(x+0) = f(x)f(0) = f(x)\cdot 0 = 0$, which is not allowed. So $f(0) = 1$.

2. The expression $1 + i + i^2 + i^3 + \cdots + i^{100}$, where $i = \sqrt{-1}$, is equal to

(a) 1 **(b)** 0 **(c)** 25 **(d)** $25 - 25i$ **(e)** None of the above

SOLUTION **(a)**: For any complex number $x \neq 1$, we have $1 + x + x^2 + \cdots + x^n = \frac{x^{n+1}-1}{x-1}$, so $1 + i + i^2 + \cdots + i^{100} = \frac{i^{101}-1}{i-1} = \frac{i-1}{i-1} = 1$.
Another solution: In the given sum, each term can be obtained from the previous term by multiplying by i. For any complex number z, the number $z\cdot i$ has the same modulus as z, and is rotated by right angle counterclockwise with respect to z. Therefore $z+z\cdot i+z\cdot i^2+z\cdot i^3 = 0$. The sum has 101 terms. It can be written as $1+(i+i^2+i^3+i^4)+(i^5+i^6+i^7+i^8)+\cdots+(i^{97}+i^{98}+i^{99}+i^{100}) = 1+0+0+0+\cdots$.

3. An arithmetic sequence is one in which each term after the first can be found by adding a constant to the preceeding term. That is, there is a fixed number d such that $a_{n+1} = a_n + d$ for all $n \geq 1$. If the first term of a finite arithmetic sequence is 6 and the last term is 101 and the sum of all the terms is 1070, what are the first 3 terms?

(a) 6, 26, 46 **(b)** 6, 53.5, 101 **(c)** 6, 10.75,15.5

(d) 6, 11, 16 **(e)** None of the above

SOLUTION **(d)**: The sum of a finite arithmetic sequence with n terms is given by $a_1 + a_2 + \cdots + a_n = \frac{n(a_1+a_n)}{2}$. In our case, this is $\frac{n(6+101)}{2} = \frac{107n}{2}$. This is supposed to be equal to 1070, so $n = 20$.
Also, the formula for n-th term of arithmetic series is $a_n = a_1 + (n-1)d$, so $101 = 6 + 19d$, and so $d = 95/19 = 5$. So the first 3 terms are 6, 11, 16.

4. James took a test with 3 parts. He got 20% of the 15 multiple choice questions right and 85% of the 20 short answer questions right. If he got 75% of the entire test right, what percent of the n vocabulary questions did he get right?

(a) 100 **(b)** $\frac{3n+25}{4}$ **(c)** $\frac{20+.75n}{35+n}$ **(d)** $\frac{625}{n}+75$ **(e)** None of the above

SOLUTION **(d)**: James got right $.2(15)+.85(20)=3+17=20$ out of $15+20=35$ from the first two parts. If x is the amount he gets right from the third part, the equation is: $\frac{20+x}{35+n}=\frac{3}{4}$. Solving for x is terms of n gives $4x=25+3n$ or $x=6.25+.75n$. The percent he gets right is

$$\frac{x}{n}\times 100=\frac{(6.25+.75n)\,100}{n}=\frac{625}{n}+75$$

5. Simplify $3(a^2+1)^2+2(a-1)(a^2+1)-5(a-1)^2-4(0.75a^4+3a-1)$.

(a) $2a^3-a^2$ **(b)** $2a^2-a^3$ **(c)** $2a^3$ **(d)** $2a^2$ **(e)** None of the above

SOLUTION **(a)**:

$$\begin{aligned}&3(a^2+1)^2+2(a-1)(a^2+1)-5(a-1)^2-4(0.75a^4+3a-1)\\ &=3(a^4+2a^2+1)+2(a^3-a^2+a-1)-5(a^2-2a+1)-3a^4-12a+4\\ &=3a^4+6a^2+3+2a^3-2a^2+2a-2-5a^2+10a-5-3a^4-12a+4\\ &=2a^3-a^2.\end{aligned}$$

6. If $x+y=a$ and $x^2+y^2=b$, express x^3+y^3 in terms of a and b.

(a) ab **(b)** a^2+b **(c)** $a+b^2$

(d) $(3ab-a^3)/2$ **(e)** None of the above

SOLUTION **(d)**: Using the sum of two cubes formula, $x^3+y^3=(x+y)(x^2-xy+y^2)=(x+y)(x^2+y^2-xy)$. From $(x+y)^2=x^2+2xy+y^2$ we obtain

$$xy=\frac{(x+y)^2-(x^2+y^2)}{2}=\frac{a^2-b}{2}$$

and

$$x^3+y^3=a\left(b-\frac{a^2-b}{2}\right)=\frac{3ab-a^3}{2}.$$

7. Simplify

$$\frac{1}{1+\frac{x}{1-\frac{x}{x+2}}} \div \frac{\frac{1}{1-x}+\frac{1}{1+x}}{\frac{1}{1-x}-\frac{1}{1+x}}$$

(a) $\frac{2x}{x^2+2x+2}$ **(b)** $\frac{2x}{x^2-1}$ **(c)** 1 **(d)** x^2-1 **(e)** None of the above

SOLUTION **(a)**:

$$\frac{1}{1+\frac{x}{1-\frac{x}{x+2}}} = \frac{1}{1+\frac{x}{\frac{2}{x+2}}} = \frac{1}{1+\frac{x^2+2x}{2}} = \frac{1}{\frac{x^2+2x+2}{2}} = \frac{2}{x^2+2x+2}$$

and

$$\frac{1}{1+\frac{x}{1-\frac{x}{x+2}}} \div \frac{\frac{1}{1-x}+\frac{1}{1+x}}{\frac{1}{1-x}-\frac{1}{1+x}} = \frac{\frac{2}{x^2-1}}{\frac{2x}{x^2-1}} = \frac{1}{x}$$

so

$$\frac{1}{1+\frac{x}{1-\frac{x}{x+2}}} \div \frac{\frac{1}{1-x}+\frac{1}{1+x}}{\frac{1}{1-x}-\frac{1}{1+x}} = \frac{2x}{x^2+2x+2}$$

8. A large school district has three high schools, Darth Vader High, Darth Sidius High, and Yoda High. In the district, 55% of the students are male, and 2,175 students attend Darth Vader High School. The males are distributed through the Darth Vader High, Darth Sidius High, and Yoda High in the ratio 1 : 1 : 3. The females are distributed in the ratio 2 : 1 : 2. How many students attend Yoda High?

   (a) 3,625 **(b)** 3,700 **(c)** 3,750 **(d)** 3,800 **(e)** 3,825

   SOLUTION **(e)**: Let x and y be the numbers of male and female students, respectively, at Darth Sidius High. Then there are x males and $2y$ females at Darth Vader High, and $3x$ males and $2y$ females at Yoda High. The total number of male students in the district is $5x$, the total number of female students is $5y$. Since 55% of the students are male,

$$\frac{5x}{5x+5y} = \frac{55}{100}$$

   which simplifies to $9x = 11y$.
   On the other hand, since 2,175 students attend Darth Vader High, we get $x + 2y = 2,175$. To obtain the values of x and y we need to solve the system

$$\begin{aligned} x + 2y &= 2175 \\ 9x - 11y &= 0 \end{aligned}$$

   Multiplying the first equation by 11 and the second by 2 and adding them together, we obtain

$$29x = 2175 * 11$$

   or $x = 75 * 11 = 825$, and $2y = 2175 - 825 = 1350$.
   Then the number of students in Yoda High is $3x + 2y = 3 * 825 + 1350 = 3,825$.

9. An auto insurance company has $10,000$ policy holders. Each policy holder is classified as:

   1. young or old

   2. male or female

   3. married or single.

   Of these policy holders, $3,000$ are young, $4,600$ are male, and $7,000$ are married. Among the policyholders there are exactly $1,320$ young males, $3,010$ married males, and $1,400$ young married persons. Finally, 600 of the policyholders are young married males.
   How many of the company's policyholders are old married females?

   (a) $3,120$ **(b)** $3,190$ **(c)** $3,220$ **(d)** $3,290$ **(e)** $3,390$

   SOLUTION **(b)**: Since there are $3,000$ young policy holders, there must be $7,000$ old policy holders. Since there are $4,600$ males, there must be $5,400$ females. Finally, since $7,000$ are married, the other $3,000$ are single.
   Since there are $1,320$ young males, there are $3,000 - 1,320 = 1,680$ young females, $4,600 - 1,320 = 3,280$ old males, and $7,000 - 3,280 = 3,720$ old females.
   Similarly, since $3,010$ are married males, $4,600 - 3,010 = 1,590$ are single makes, $7,000 - 3,010 = 3,990$ are married females, and $5,400 - 3,990 = 1,410$ single females.
   Also, since $1,400$ policyholders are young and married, $3,000 - 1,400 = 1,600$ are young and single, $7,000 - 1,400 = 5,600$ are old and married, and $3,000 - 1,600 = 1,400$ are old and single.
   Finally, looking only at the married policyholders, out of the $3,010$ married males, 600 are young, and $3,010 - 600 = 2,410$ are old. That means that, out of the $5,600$ married old people, $2,410$ are males, and so $5,600 - 2,410 = 3,190$ are old married females.

10. For what values of a will the following equation have no solution:

$$\frac{1}{x+(a-1)} - \frac{2a}{x^2-(a-1)^2} = \frac{5}{x-(a-1)}$$

(a) $a = \frac{1}{2}$ and $a = \frac{5}{6}$ **(b)** $a = \frac{1}{2}$ and $a = \frac{5}{2}$ **(c)** $a = \frac{5}{6}$ and $a = \frac{5}{2}$ **(d)** $a = \frac{1}{2}$

(e) $a = \frac{5}{2}$

Solution **(a)**: The equation will have no solution of $x = \pm(a-1)$. To solve the equation for x in terms of a, multiply both sides by $[x-(a-1)][x+(a-1)]$ (assuming $x \neq \pm(a-1)$) to obtain

$$\begin{aligned}(x-(a-1)) - 2a &= 5(x+(a-1))\\ x - a + 1 - 2a &= 5x + 5a - 5\\ -4x &= 8a - 6\\ x &= \frac{-4a+3}{2}\end{aligned}$$

So the equation will have no solution if $\frac{-4a+3}{2} = \pm(a-1)$ which will happen if $a = \frac{1}{2}$ and if $a = \frac{5}{6}$.

11. Determine all values of k so that the equation

$$kx^2 - 2x + 4 - 4k = 0$$

has two solutions, one of which is negative and the other is positive.

(a) $k < 0$ **(b)** $k > 1$ **(c)** $k < 0$ or $k > 1$ **(d)** $k \neq 0$

(e) $k \neq 0$ and $k < 2$

Solution **(c)**: If $k = 0$, the equation is linear and has only one solution. Provided $k \neq 0$, the solutions of the equation are

$$x = \frac{2 \pm \sqrt{4-4k(4-4k)}}{2k} = \frac{2 \pm \sqrt{4-16k+16k^2}}{2k} = \frac{2 \pm \sqrt{(2-4k)^2}}{2k} = \frac{2 \pm (2-4k)}{2k}$$

The first solution is

$$x_1 = \frac{2+2-4k}{2k} = \frac{2(1-k)}{k},$$

the second solution is

$$x_2 = \frac{2-(2-4k)}{2k} = \frac{4k}{2k} = 2.$$

We need $x_1 < 0$, i.e. $\frac{2(1-k)}{k} < 0$, which will happen if $k < 0$ or $k > 1$.

12. Find the remainder when $3^{888,888}$ is divided by 5.

(a) 1.4 **(b)** 1 **(c)** 6 **(d)** 5 **(e)** 0

SOLUTION **(b)**: We have $3^1 = 3$ (remainder $R = 3$), $3^2 = 9$ (remainder $R = 4$), $3^3 = 27$ (remainder $R = 2$), $3^4 = 81$ (remainder $R = 1$), and after that, the remainders of the powers of 3 when divided by 5 start to repeat. In general, for a positive integer n, the remainder R when 3^n is divided by 5 depends upon the remainder r when n is divided by 4, as follows: if $r = 1$, then $R = 3$; if $r = 2$, then $R = 4$; if $r = 3$, then $R = 2$; if $r = 0$, then $R = 1$. Now $888,888$ has remainder 0 when divided by 4, so $3^{888,888}$ has remainder 1 when divided by 5.

13. Joe has three drinking cups: a cylinder-shaped, a cone-shaped and a semi-sphere-shaped. All of them have the same radius and height. The cylinder-shaped cup is full of water. Joe want to pour the water from the cylinder-shaped cup to the other two cups. Which of the following is going to happen?

(a) There will not be enough water to fill the two cups.

(b) The water from the cylinder-shaped cup will exactly fill the two other cups.

(c) There will be too much water to fit into the other two cups.

(d) It depends on what the radius and height are.

(e) It depends on what the radius is.

SOLUTION **(b)**: Because the semi-spherical cup must have a height h equal to its radius r, all 3 cups must have height r and radius r. The volumes of the cylinder-shaped, cone-shaped, and semi-sphere-shaped cups are, respectively, $\pi r^2 h = \pi r^3$, $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3$, and $\frac{1}{2}\frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$ cubic units. So the water in the cylindrical-shaped cup will exactly fill the other two cups.

14. If A is a 3×3 matrix such that

$$A\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 & 8 \\ 4 & 1 & 4 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

what is the value in the third row, second column of A?

(a) 0 **(b)** 1 **(c)** 2 **(d)** 4 **(e)** None of the above

SOLUTION **(b)**: Let $[a\ b\ c]$ be the third row of A. Then we have $a + 2c = 0$, $2a + b = 1$, $3a + 2c = 0$, and $4a + b = 1$. Subtracting the first equation from the third, we find $2a = 0$, so $a = 0$. Therefore, $c = 0$, so $b = 1$.

15. Suppose a group of people participate in a messaging system in which they can send messages to others in the group. The system works according the following two rules:

   - If A can send a message to B and B can send a message to C, then C can send a message to A.
   - For each pair of distinct people A and B in the group, either A can send a message to B or B can send a message to A but not both.

   Which of the following is true:

   (a) There cannot exist a messaging system satisfying these two rules.

   (b) There must be an even number of people in the group.

   (c) There must be an odd number of people in the group.

   (d) The group can have no more than 3 members.

   (e) There is no restriction on the number of people in the group.

   SOLUTION **(d)**: We could have exactly two people A and B in the group, where A can send a message to B, but B cannot send a message to A. We could also have exactly three people A, B, and C, where A can message B, B can message C, and C can message A, with no other ways to send messages. So there do exist messaging systems satisfying these two rules, and the number of people in the group may be even or odd.

   Now suppose that a, b, and c are three distinct people in a group with a communication system satisfying these two rules. If c can send a message to both a and b, then a must not be able to send a message to b, since that would imply that b could also send a message to c; and similarly, b must not be able to send a message to a, since then a could send a message to c. Since a and b must be able to send a message to each other in some order, it is therefore impossible for c to be able to send a message to both a and b. This argument shows that each person can send a message to at most one other person in the group, and therefore that any three people a, b, c must be able to send messages only in one of the two patterns $a \to b \to c \to a$ or $a \to c \to b \to a$.

   If there were a fourth person d in the group, then d would only be able to send a message to at most one of a, b, and c. But by considering the set of d together with the other two people, we see from the paragraph above that this is impossible. Therefore, the group must contain at most 3 people.

16. For which values of k does the system

$$x^2 - y^2 = 0$$
$$(x-k)^2 + y^2 = 2$$

have exactly two distinct solutions in the form (x, y) where both x and y are real numbers?

(a) $k = \pm 1$ **(b)** $k = \pm 2$ **(c)** $k = 1$ and $k = -2$

(d) $k = -1$ and $k = 2$ **(e)** None of the above

Solution **(b)**: If (x, y) is a solution, then so is $(x, -y)$. The system is equivalent to $y = \pm x$, $(x-k)^2 + x^2 = 2$. The quadratic equation $(x-k)^2 + x^2 = 2$ has discriminant $4k^2 - 8(k^2 - 2) = 4(4 - k^2)$. When the discriminant is 0, we find, for $k = 2$, the solution set is $\{(1, 1), (1, -1)\}$, and when $k = -2$, it is $\{(-1, 1), (-1, -1)\}$. When the discriminant is negative, there are no real solutions x to the quadratic equation, hence no real solutions (x, y) to the system. When the discriminant is positive, there are two distinct real solutions x_1, x_2 to the quadratic equation, at least one of which must be non-zero; hence, the solutions to the system are $(x_1, \pm x_1)$, $(x_2, \pm x_2)$ in this case, and these solutions are at least 3 in number. Therefore, there are exactly 2 values of k for which the system has exactly two distinct solutions, namely, $k = \pm 2$.

17. Find the number of pairs of positive integers that have greatest common divisor 3! and the least common multiple 18!.

(a) 2 **(b)** 2^6 **(c)** 2^7 **(d)** 2^{18} **(e)** None of the above

Solution **(b)**:

$$3! = 2 \cdot 3$$
$$18! = 2^{16} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$$

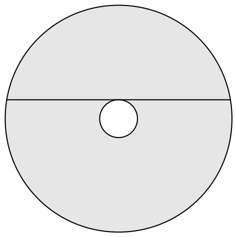
Both integers must have 2 and 3 as factors, and one of the integers must have no higher power of 2, and one of the integers must have no higher power of 3.
I addition to the factors 2 and 3, the factors 2^{15}, 3^7, 5^3, 7^2, 11, 13 and 17 have to be divided among the two integers, such that each of the factors appears in exactly one of the two integers. Since there are 7 such factors, there are 2^7 ways to distribute these factors among the two integers. However, for each such way there will always be another way that will produce the same pair of integers. Therefore there are 2^6 distinct pairs of integers satisfying the given conditions.

18. If $\log_y x + \log_x y = 7$, then what is the value of $\left(\log_y x\right)^2 + (\log_x y)^2$?

(a) 40 **(b)** 43 **(c)** 45 **(d)** 47 **(e)** 49

Solution **(d)**: We have $\log_y x = \frac{\ln x}{\ln y}$ and $\log_x y = \frac{ln y}{\ln x}$, so $(\log_y x)(\log_x y) = 1$. Thus $(\log_y x + \log_x y)^2 = 7^2 = 49$ gives $(\log_y x)^2 + (\log_x y)^2 + 2 = 49$, so $(\log_y x)^2 + (\log_x y)^2 = 47$.

19. The circles in the figure shown are concentric. The chord shown is tangent to the inner circle and has length 12. What is the area of the shaded region?



(a) 24π **(b)** 32π **(c)** 36π **(d)** 40π **(e)** 48π

SOLUTION **(c)**: Let r and R be the radii of the small and the large circle, respectively. The area of the shaded region is $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$. Since the chord is tangent to the inner circle, the triangle in the picture on the left is a right triangle, and from the Pythagorean Theorem, $R^2 = r^2 + 6^2$, and so $R^2 - r^2 = 36$. So the area of the shaded region is 36π.

20. How many 5-digit numbers with all digits non-zero and no digit repeated are divisible by 25?

(a) 360 **(b)** 420 **(c)** 450 **(d)** 480 **(e)** 500

SOLUTION **(b)**: In base ten, a number is divisible by 25 if and only if its last two digits are 00, 25, 50, or 75. The first and third cases cannot happen here, since we require our digits to be non-zero. The remaining two cases leave 7 digits left to fill in the first three digits of our number, and there are $7 \times 6 \times 5 = 210$ ways to complete the number in each case. So there are a total of $2 \times 210 = 420$ ways to choose such a number.

21. Each of two boxes contains 20 marbles, and each marble is either black or white. The total number of black marbles is different from the total number of white marbles. One marble is drawn at random from each box. The probability that both marbles are white is 0.21. What is the probability that both are black?

(a) 0.22 **(b)** 0.23 **(c)** 0.24 **(d)** 0.25 **(e)** 0.26

SOLUTION **(e)**: Let w_i be the number of white marbles in box $\#i$ for $i = 1, 2$. The probability of drawing two white marbles is $\frac{w_1}{20} \times \frac{w_2}{20} = 0.21$, so $w_1 w_2 = 84 = 2^2 \cdot 3 \cdot 7$. So one of the w_i is divisible by 7; say w_1 is divisible by 7. Then $w_1 = 7, w_2 = 12$ or $w_1 = 14, w_2 = 6$. In the second case, the total number of white marbles is 20, so the total number of black marbles must also be 20, which contradicts our assumptions. Therefore, $w_1 = 7$ and $w_2 = 12$. The probability that both marbles are black is $\frac{20-w_1}{20} \cdot \frac{20-w_2}{20} = \frac{13}{20} \cdot \frac{8}{20} = \frac{26}{100} = 0.26$.

22. If the repeating decimal $0.84\overline{51}$ is represented by the fraction $\frac{a}{b}$, where a and b are positive integers with no common factors greater than 1, find $a + b$.

(a) 303 **(b)** 4617 **(c)** 5211 **(d)** 6089 **(e)** 8451

SOLUTION **(d)**: $0.84\overline{51} = 0.84 + 51 \times 0.0001 \times 1.\overline{01}$. Also, $1.\overline{01} = 1 + x + x^2 + \cdots$, where $x = 0.01$ so $1.\overline{01} = \frac{1}{1-0.01} = \frac{100}{99}$. Thus, $0.84\overline{51} = 0.84 + 51 \times 0.0001 \times \frac{100}{99} = 0.84 + \frac{0.51}{99} = \frac{84}{100} + \frac{51}{9900} = \frac{84\times99+51}{9900} = \frac{84\times33+17}{3300} = \frac{2772+17}{3300} = \frac{2789}{3300}$. The only divisors of 3300 greater than 1 are 2, 3, 5, and 11, none of which divides 2789, so this fraction is in lowest terms. So $a = 2789$, $b = 3300$, and $a + b = 6089$.

23. If $f(x) = \frac{a}{x-4}$ and $g(x) = \frac{b}{x}$, and if $(f \circ g)(-1) = -\frac{1}{2}$ and $(f \circ g)(1) = -\frac{3}{2}$, find $a + 3b$.

(a) 6 **(b)** 7 **(c)** 8 **(d)** 9 **(e)** 10

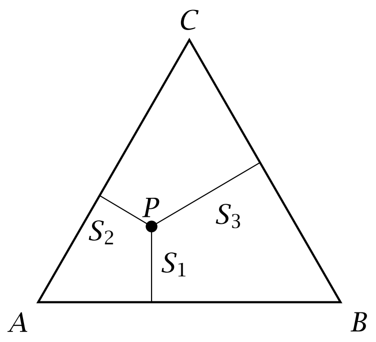
SOLUTION **(d)**: We have $g(-1) = -b$, $g(1) = b$, so $f(g(-1)) = \frac{a}{-b-4} = -\frac{1}{2}$ and $f(g(1)) = \frac{a}{b-4} = -\frac{3}{2}$. Dividing these equations, we get $\frac{-b-4}{b-4} = 3$, so $-b - 4 = 3b - 12$ and $b = 2$. Substituting $b = 2$ in one of the previous equations gives $a = 3$. Therefore, $a + 3b = 3 + 3 \cdot 2 = 9$.

24. If $\sin x = 2\cos x$, find the value of $\sin x \cos x$.

(a) $\frac{1}{3}$ **(b)** $\frac{2}{3}$ **(c)** $\frac{1}{4}$ **(d)** $\frac{1}{5}$ **(e)** $\frac{2}{5}$

SOLUTION **(e)**: Let T be a right triangle with hypotenuse of length 1 and interior angle x. Let a and b be the adjacent and opposite side lengths, respectively. Then $b = \sin(x) = 2\cos(x) = 2a$, and $a^2 + b^2 = 1$. So $5a^2 = 1$ and $a^2 = \frac{1}{5}$. Now $\sin(x)\cos(x) = ab = 2a^2 = \frac{2}{5}$.

25. Given equilateral triangle ABC with $AB = 2$ and a point P in the interior of ABC. Let S_1, S_2 and S_3 be the perpendicular distances from P to each side of the triangle ABC. What is $S_1 + S_2 + S_3$?



(a) 2

(b) $\sqrt{2}$

(c) $\sqrt{3}$

(d) 3 **(e)** Not enough information given

SOLUTION **(c)**: Draw line segments from P to each vertex A, B, and C. Let $a, 2-a, b, 2-b, c, 2-c$ be the lengths of the legs of the resulting six small triangles along the sides of the triangle ABC. The sum of the areas of the six small triangles is $\frac{1}{2}aS_1 + \frac{1}{2}(2-a)S_1 + \frac{1}{2}bS_2 + \frac{1}{2}(2-b)S_2 + \frac{1}{2}cS_3 + \frac{1}{2}(2-c)S_3 = S_1 + S_2 + S_3$. Also, this sum must be the area of triangle ABC. The height of triangle ABC can be found by dropping a perpendicular from C to side AB, which bisects AB, so that $1 + h^2 = 2^2$ and $h = \sqrt{3}$. Therefore, $S_1 + S_2 + S_3 = \sqrt{3}$.