Saginaw Valley State University

2010 Math Olympics – Level II

1. If f is a nonzero function (this means there is at least one x with $f(x) \neq 0$) of real numbers such that $f(x+y) = f(x)f(y)$, what are the possible values for $f(0)$?

   (a) any real number is possible **(b)** any positive real number is possible

   (c) $f(0)$ must be 0 **(d)** $f(0)$ must be 1

   (e) $f(0)$ could be 0 or 1

2. The expression $1 + i + i^2 + i^3 + \cdots + i^{100}$, where $i = \sqrt{-1}$, is equal to

   (a) 1 **(b)** 0 **(c)** 25 **(d)** $25 - 25i$ **(e)** None of the above

3. An arithmetic sequence is one in which each term after the first can be found by adding a constant to the preceeding term. That is, there is a fixed number d such that $a_{n+1} = a_n + d$ for all $n \geq 1$. If the first term of a finite arithmetic sequence is 6 and the last term is 101 and the sum of all the terms is 1070, what are the first 3 terms?

   (a) 6, 26, 46 **(b)** 6, 53.5, 101 **(c)** 6, 10.75,15.5

   (d) 6, 11, 16 **(e)** None of the above

4. James took a test with 3 parts. He got 20% of the 15 multiple choice questions right and 85% of the 20 short answer questions right. If he got 75% of the entire test right, what percent of the n vocabulary questions did he get right?

   (a) 100 **(b)** $\frac{3n+25}{4}$ **(c)** $\frac{20+.75n}{35+n}$ **(d)** $\frac{625}{n} + 75$ **(e)** None of the above

5. Simplify $3(a^2+1)^2 + 2(a-1)(a^2+1) - 5(a-1)^2 - 4(0.75a^4 + 3a - 1)$.

   (a) $2a^3 - a^2$ **(b)** $2a^2 - a^3$ **(c)** $2a^3$ **(d)** $2a^2$ **(e)** None of the above

6. If $x + y = a$ and $x^2 + y^2 = b$, express $x^3 + y^3$ in terms of a and b.

   (a) ab **(b)** $a^2 + b$ **(c)** $a + b^2$

   (d) $(3ab - a^3)/2$ **(e)** None of the above

7. Simplify

$$\frac{1}{1+\frac{x}{1-\frac{x}{x+2}}} \div \frac{\frac{1}{1-x}+\frac{1}{1+x}}{\frac{1}{1-x}-\frac{1}{1+x}}$$

(a) $\frac{2x}{x^2+2x+2}$ **(b)** $\frac{2x}{x^2-1}$ **(c)** 1 **(d)** x^2-1 **(e)** None of the above

8. A large school district has three high schools, Darth Vader High, Darth Sidius High, and Yoda High. In the district, 55% of the students are male, and $2,175$ students attend Darth Vader High School. The males are distributed through the Darth Vader High, Darth Sidius High, and Yoda High in the ratio $1:1:3$. The females are distributed in the ratio $2:1:2$. How many students attend Yoda High?

(a) $3,625$ **(b)** $3,700$ **(c)** $3,750$ **(d)** $3,800$ **(e)** $3,825$

9. An auto insurance company has $10,000$ policy holders. Each policy holder is classified as:

   1. young or old

   2. male or female

   3. married or single.

   Of these policy holders, $3,000$ are young, $4,600$ are male, and $7,000$ are married. Among the policyholders there are exactly $1,320$ young males, $3,010$ married males, and $1,400$ young married persons. Finally, 600 of the policyholders are young married males.
   How many of the company's policyholders are old married females?

(a) $3,120$ **(b)** $3,190$ **(c)** $3,220$ **(d)** $3,290$ **(e)** $3,390$

10. For what values of a will the following equation have no solution:

$$\frac{1}{x+(a-1)} - \frac{2a}{x^2-(a-1)^2} = \frac{5}{x-(a-1)}$$

(a) $a=\frac{1}{2}$ and $a=\frac{5}{6}$ **(b)** $a=\frac{1}{2}$ and $a=\frac{5}{2}$ **(c)** $a=\frac{5}{6}$ and $a=\frac{5}{2}$ **(d)** $a=\frac{1}{2}$

(e) $a=\frac{5}{2}$

11. Determine all values of k so that the equation

$$kx^2 - 2x + 4 - 4k = 0$$

has two solutions, one of which is negative and the other is positive.

(a) $k < 0$ **(b)** $k > 1$ **(c)** $k < 0$ or $k > 1$ **(d)** $k \neq 0$

(e) $k \neq 0$ and $k < 2$

12. Find the remainder when $3^{888,888}$ is divided by 5.

(a) 1.4 **(b)** 1 **(c)** 6 **(d)** 5 **(e)** 0

13. Joe has three drinking cups: a cylinder-shaped, a cone-shaped and a semi-sphere-shaped. All of them have the same radius and height. The cylinder-shaped cup is full of water. Joe want to pour the water from the cylinder-shaped cup to the other two cups. Which of the following is going to happen?

(a) There will not be enough water to fill the two cups.

(b) The water from the cylinder-shaped cup will exactly fill the two other cups.

(c) There will be too much water to fit into the other two cups.

(d) It depends on what the radius and height are.

(e) It depends on what the radius is.

14. If A is a 3×3 matrix such that

$$A\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 & 8 \\ 4 & 1 & 4 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

what is the value in the third row, second column of A?

(a) 0 **(b)** 1 **(c)** 2 **(d)** 4 **(e)** None of the above

15. Suppose a group of people participate in a messaging system in which they can send messages to others in the group. The system works according the following two rules:

    – If A can send a message to B and B can send a message to C, then C can send a message to A.

    – For each pair of distinct people A and B in the group, either A can send a message to B or B can send a message to A but not both.

    Which of the following is true:

    (a) There cannot exist a messaging system satisfying these two rules.

    (b) There must be an even number of people in the group.

    (c) There must be an odd number of people in the group.

    (d) The group can have no more than 3 members.

    (e) There is no restriction on the number of people in the group.

16. For which values of k does the system

$$x^2 - y^2 = 0$$
$$(x-k)^2 + y^2 = 2$$

    have exactly two distinct solutions in the form (x, y) where both x and y are real numbers?

    (a) $k = \pm 1$ **(b)** $k = \pm 2$ **(c)** $k = 1$ and $k = -2$

    (d) $k = -1$ and $k = 2$ **(e)** None of the above

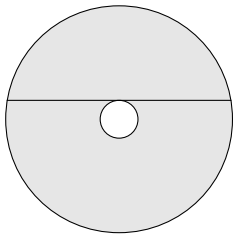
17. Find the number of pairs of positive integers that have greatest common divisor 3! and the least common multiple 18!.

    (a) 2 **(b)** 2^6 **(c)** 2^7 **(d)** 2^{18} **(e)** None of the above

18. If $\log_y x + \log_x y = 7$, then what is the value of $\left(\log_y x\right)^2 + (\log_x y)^2$?

    (a) 40 **(b)** 43 **(c)** 45 **(d)** 47 **(e)** 49

19. The circles in the figure shown are concentric. The chord shown is tangent to the inner circle and has length 12. What is the area of the shaded region?



(a) 24π **(b)** 32π **(c)** 36π **(d)** 40π **(e)** 48π

20. How many 5-digit numbers with all digits non-zero and no digit repeated are divisible by 25?

(a) 360 **(b)** 420 **(c)** 450 **(d)** 480 **(e)** 500

21. Each of two boxes contains 20 marbles, and each marble is either black or white. The total number of black marbles is different from the total number of white marbles. One marble is drawn at random from each box. The probability that both marbles are white is 0.21. What is the probability that both are black?

(a) 0.22 **(b)** 0.23 **(c)** 0.24 **(d)** 0.25 **(e)** 0.26

22. If the repeating decimal $0.84\overline{51}$ is represented by the fraction $\frac{a}{b}$, where a and b are positive integers with no common factors greater than 1, find $a+b$.

(a) 303 **(b)** 4617 **(c)** 5211 **(d)** 6089 **(e)** 8451

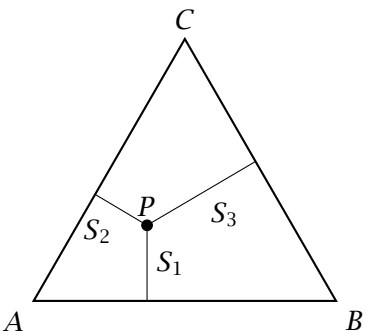
23. If $f(x)=\frac{a}{x-4}$ and $g(x)=\frac{b}{x}$, and if $(f\circ g)(-1)=-\frac{1}{2}$ and $(f\circ g)(1)=-\frac{3}{2}$, find $a+3b$.

(a) 6 **(b)** 7 **(c)** 8 **(d)** 9 **(e)** 10

24. If $\sin x = 2\cos x$, find the value of $\sin x \cos x$.

(a) $\frac{1}{3}$ **(b)** $\frac{2}{3}$ **(c)** $\frac{1}{4}$ **(d)** $\frac{1}{5}$ **(e)** $\frac{2}{5}$

25. Given equilateral triangle ABC with $AB = 2$ and a point P in the interior of ABC. Let S_1, S_2 and S_3 be the perpendicular distances from P to each side of the triangle ABC. What is $S_1+S_2+S_3$?



(a) 2

(b) $\sqrt{2}$

(c) $\sqrt{3}$

(d) 3 **(e)** Not enough information given