Saginaw Valley State University
2010 Math Olympics – Level I Solutions

1. For positive real numbers x, y, and z, which of the following is equivalent to $x^{-\frac{1}{2}}y^{\frac{2}{3}}z^{\frac{3}{5}}$?

(a) $\sqrt[30]{(xyz)^{23}}$ **(b)** $x^2\sqrt[15]{y^{10}z^9}$ **(c)** $\sqrt[30]{x^{-1}y^2z^3}$

(d) $\frac{\sqrt[30]{x^{15}y^{20}z^{18}}}{x}$ **(e)** None of the above

SOLUTION **(d)**: Rewriting all exponents with the common denominator 30 will give us

$$x^{-\frac{15}{30}}y^{\frac{20}{30}}z^{\frac{18}{30}} = \frac{\sqrt[30]{y^{20}z^{18}}}{\sqrt[30]{x^{15}}}.$$

Rationalizing the denominator by multiplying both the numerator and denominator by $\sqrt[30]{x^{15}}$ will give us

$$\frac{\sqrt[30]{x^{15}y^{20}z^{18}}}{x}.$$

2. If f is a nonzero function (this means there is at least one x with $f(x) \neq 0$) of real numbers such that $f(x+y) = f(x)f(y)$, what are the possible values for $f(0)$?

(a) any real number is possible **(b)** any positive real number is possible

(c) $f(0)$ must be 0 **(d)** $f(0)$ must be 1

(e) $f(0)$ could be 0 or 1

SOLUTION **(d)**: Setting $y = 0$ in the identity $f(x+y) = f(x)f(y)$ will give us $f(x) = f(x)f(0)$. Since f is a nonzero function, there must be at least one x such that $f(x) \neq 0$. We can then divide by $f(x)$ and obtain $1 = f(0)$.

3. Find a so that the quadratic equation $(a-1)x^2+3x-a+2=0$ has a solution $x=-1$.

(a) No such a **(b)** $a=0$ **(c)** $a=1$ **(d)** $a=2$

(e) None of the above

SOLUTION **(a)**: Let x_1 and x_2 be two solutions of the quadratic equation, and let $x_2=-1$. Then $x_1+x_2=\frac{-3}{a-1}$ and so $x_1=\frac{a-4}{a-1}$.
On the other hand, $x_1x_2=\frac{2-a}{a-1}$ and so $x_1=\frac{a-2}{a-1}$. Since the equation

$$\frac{a-4}{a-1}=\frac{a-2}{a-1}$$

has no solution, there cannot be any such a.
Another solution: Assume there is such a. Plug in -1 for x to obtain $(a-1)(-1)^2+3(-1)-a+2=0$, or $a-1-3-a+2=0$, or $-2=0$, which is a contradiction.

4. A group of 50 high school math concentration students are comparing how many of their three different math exams (Algebra, Geometry and Calculus) they passed at the first attempt. You are given the following information:

1. 56% of students passed Geometry at the first attempt.

2. 58% of students passed Calculus at the first attempt.

3. 19 students passed both Geometry and Calculus at the first attempt.

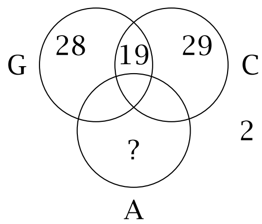
4. 2 students did not pass any exam at the first attempt.

Calculate how many students passed only the Algebra exam at the first attempt.

(a) 3 **(b)** 5 **(c)** 10

(d) 24 **(e)** Not enough information

SOLUTION **(c)**:
From the given data, we know that 28 students passed Geometry at the first attempt, and 29 students passed Calculus at the first attempt. There were 19 students that passed *both* Geometry and Calculus at the first attempt. So the number of students who passed Geometry or Calculus at the first attempt was $28+29-19=38$. Since 2 students did not pass any exam at the first attempt, we have 48 students that passed at least one exam at the first attempt. Out of these 48 students, 38 passed Geometry of Calculus at the first attempt. That leaves us with $48-38=10$ students who passes only Algebra at the first attempt.



5. The Bathula family has 6 sons. Each son has 3 sisters. How many children are there?

(a) 37 **(b)** 9 **(c)** 36 **(d)** 18 **(e)** 24

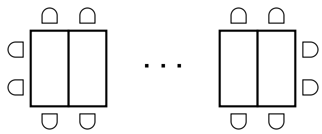
SOLUTION **(b)**: There are 6 boys and 3 girls, altogether 9 children.

6. A snail at the bottom of a well goes up 10 feet each day and slides back 5 feet at night. How many days does it take the snail reach the top of the well if it is 40 feet deep?

(a) 4 **(b)** 5 **(c)** 3 **(d)** 8 **(e)** 7

SOLUTION **(e)**: The snail will climb 10 feet up and slides 5 feet down every day except the last day. On the last day, the snail will reach the top and leave the well, which means that on this last day, the snail will only climb, not slide. On the last day the snail will therefore gain 10 feet, while on each of the previous days the snail will gain only 5 feet. The snail has to climb 10 feet during the last day, and the remaining 30 feet during the rest of the days. It will take the snail $30/5 = 6$ days to climb the 30 feet, so altogether it will take 7 days to reach the top of the well.

7. I am inviting some people to a party, and I want to seat them all at one long table. I want to put together a series of rectangular tables, and form one long table, as shown. What is a formula that relates the number of chairs (C) to the number of tables (T)?



(a) $C = 2T + 4$ **(b)** $C = 2T$ **(c)** $C = 4T + 2$ **(d)** $C = T + 4$ **(e)** $C = T$

SOLUTION **(a)**: Each of the rectangular tables will have a chair at each of its shorter sides, that is two chairs per table. There will be $2T$ such chairs. In addition to that, the first and the last table will also each have 2 chairs along one of its longer sides, which gives 4 extra chairs. So the total number or chairs will be $2T + 4$.

8. The set A has c members, and the set B has f members, where $c \geq f$. What is the largest number of members $A \cup B$ i.e., the set of elements that are in A or B, could have?

(a) $c - f$ **(b)** $c \cdot f$ **(c)** $c + f$ **(d)** f **(e)** c

SOLUTION **(c)**: The largest number of elements in the set $A \cup B$ will be obtained in the case in which the intersection of the two set is empty, that is, there are no elements that would be in both sets at the same time. In that case the set $A \cup B$ will contain all c elements of A and all f elements of B with no overlap. So the total number of elements will be $c + f$.

9. Find all possible ways to fill in the missing digits so that the number 546,5__ is divisible by 2 and 5 but not by 3.

(a) 10, 20, 30, 40, 50, 60, 70, 80, 90 **(b)** 00, 20, 30, 50, 60, 80

(c) 00, 10, 30, 50, 70, 90 **(d)** 00, 20, 30, 50, 60, 80, 90

(e) 20, 30, 50, 60, 80, 90

SOLUTION **(d)**: In order for the number number to be divisible by 5, the *ones* digit has to be 0 or 5. In order for the number to be divisible by 2, the *ones* digit must be even. Since 5 is odd, the *ones* digit will have to be 0. A number is divisible by 3 if and only if the sum of its digits is divisible by 3. If we call the *tens* digit t, the number will be divisible by 3 if $5+4+6+5+t = 20+t$ is divisible by 3. That will happen if $t = 1$, $t = 4$ and $t = 7$. For all the other choices of t, that is $t = 0, 2, 3, 5, 6, 8$ and 9, the number is not divisible by 3.

10. What base-eight numeral follows 37_{eight}?

(a) 37_{eight} **(b)** 47_{eight} **(c)** 38_{eight} **(d)** 100_{eight} **(e)** 40_{eight}

SOLUTION **(e)**: It cannot be 38_{eight}, since base-eight only uses digits 0 through 7. In base-eight, $7_{\text{eight}} + 1_{\text{eight}} = 10_{\text{eight}}$, so when adding 1_{eight} to 37_{eight}, we need to carry and we obtain 40_{eight}.

11. Convert base-ten numeral 15 to base eight.

(a) 16_{eight} **(b)** 10_{eight} **(c)** 12_{eight} **(d)** 17_{eight} **(e)** 13_{eight}

SOLUTION **(d)**:

$$15/8 = 1R7$$

In other words, 15 contains 1 eight and 7 ones. Therefore the base-eight form of the base-ten numeral 15 is 17_{eight}.

12. Two buses leave the terminal at 8 A.M. Bus number 36 takes 85 minutes to complete its route; bus number 86 takes 102 minutes. When is the next time the two buses will arrive together at the terminal (assuming they are on time and they spend no time waiting between routes)?

(a) in 510 min **(b)** in 102 min **(c)** in 289 min **(d)** in 408 min

(e) in 8,670 min

SOLUTION **(a)**: The number of minutes before they both arrive together at the terminal must be the least common multiple of 85 and 102.

$$85 = 5 \cdot 17$$

$$102 = 2 \cdot 3 \cdot 17$$

The least common multiple of the two numbers is then $2 \cdot 3 \cdot 5 \cdot 17 = 10 \cdot 51 = 510$.

13. Let $a = 2^5 \cdot 5^3 \cdot 11^2$. Let b be such that the greatest common factor of a and b is $2 \cdot 5^3 \cdot 11$ and the least common multiple of a and b is $2^5 \cdot 3^3 \cdot 5^3 \cdot 11^2$. Find b.

(a) $b = 2 \cdot 3^3 \cdot 5^2 \cdot 11^2$ **(b)** $b = 2^5 \cdot 3^3 \cdot 5^3 \cdot 11^2$ **(c)** $b = 2^5 \cdot 3^5 \cdot 5^6 \cdot 11$

(d) $b = 2 \cdot 3^3 \cdot 5^3 \cdot 11$ **(e)** $b = 2 \cdot 3 \cdot 5 \cdot 11$

SOLUTION **(d)**: Since $2 \cdot 5^3 \cdot 11$ is the greatest common factor of a and b, b must be equal to $2 \cdot 5^3 \cdot 11 \cdot c$ where c is not divisible by 2, 5 and 11. On the other hand, c must be a factor of the least common multiple. The only factor of the least common multiple that is not divisible by 2, 5 or 11 is 3^3. Therefore b must be 3^k for some $k \le 3$. Suppose $k < 3$. Then $2^5 \cdot 3^k \cdot 5^3 \cdot 11^2$ would be a common multiple of a and b, however, since it is less than the given least common multiple of a and b, this is a contradiction. Therefore $k = 3$, $c = 3^3$ and $b = 2 \cdot 3^3 \cdot 5^3 \cdot 11$.

Another solution: For any two positive integers a and b, $a \cdot b = LCM(a, b) \cdot GCF(a, b)$. Therefore

$$b = \frac{LCM(a,b) \cdot GCD(a,b)}{a} = \frac{2^5 \cdot 3^3 \cdot 5^3 \cdot 11^2 \cdot 2 \cdot 5^3 \cdot 11}{2^5 \cdot 5^3 \cdot 11^2} = 2 \cdot 3^3 \cdot 5^3 \cdot 11.$$

14. If a is 50% more than b, then b is how many percent of a? Please round the answer to the nearest tenth.

(a) 71.7 **(b)** 56.7 **(c)** 66.7 **(d)** 67.7 **(e)** 64.7

SOLUTION **(c)**: If a is 50% more than b, then $a = 1.5b$, or

$$b = \frac{1}{1.5}a = \frac{10}{15}a = \frac{2}{3}a \approx 0.667a$$

so b is approximately 66.7% of a.

15. If the repeating decimal $0.84\overline{51}$ is represented by the fraction $\frac{a}{b}$, where a and b are positive integers with no common factors greater than 1, find $a+b$.

(a) 303 **(b)** 4617 **(c)** 5211 **(d)** 6089 **(e)** 8451

SOLUTION **(d)**: Denote the number $0.84\overline{51}$ by x. Then $100x = 84.51\overline{51}$ and $99x = 100x - x = 84.51\overline{51} - 0.84\overline{51} = 83.67$. Then

$$x = \frac{83.67}{99} = \frac{8367}{9900} = \frac{8367}{2^2 \cdot 3^2 \cdot 5^2 \cdot 11}$$

Since 8367 is divisible by 3, but not by 2, 5, 9 and 11, the last fraction can be reduced to the simplest form

$$\frac{2789}{3300}.$$

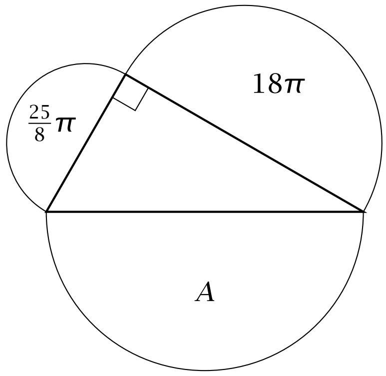
Therefore $a = 2789$ and $b = 3300$, which makes $a + b = 6089$.

16. The area of a circle is 36π m^2. What is the exact circumference?

(a) 12π m **(b)** 12 m **(c)** 6π m **(d)** 6 m **(e)** 36π m

SOLUTION **(a)**: The area of a circle is πr^2. If the area is 36π, r^2 must be 36, which makes $r = 6$. The exact circumference is $2\pi r = 2\pi 6 = 12\pi$. Since the area was given in square meters, the circumference will be in meters.

17. The areas of the two smaller semicircles are $\frac{25}{8}\pi$ and 18π. Find the area of the largest semicircle.



(a) $\frac{169}{8}\pi$ **(b)** 25π **(c)** $\frac{25}{8}\pi$ **(d)** 50π **(e)** $\frac{289}{8}\pi$

SOLUTION **(a)**: Label the shortest side of the triangle a, the second shortest side b and the longest side, the hypotenuse of the right triangle, c. The radii of the three semicircles are $a/2$, $b/2$ and $c/2$. The area of the largest semicircle is

$$\frac{1}{2}\pi\left(\frac{c}{2}\right)^2 = \frac{\pi}{8}c^2.$$

According to the Pythagorean Theorem, $c^2 = a^2 + b^2$, so the area of the largest semicircle is

$$\frac{\pi}{8}\left(a^2 + b^2\right) = \frac{\pi}{8}a^2 + \frac{\pi}{8}b^2$$

But $\frac{\pi}{8}a^2$ is the area of the smallest semicircle and $\frac{\pi}{8}b^2$ is the area of the medium semicircle. In other words, the area of the semicircle over the hypotenuse is the sum of the areas of the semicircles over the other two sides. In this case, it is

$$\frac{25}{8}\pi + 18\pi = \frac{25}{8}\pi + \frac{144}{8}\pi = \frac{169}{8}\pi.$$

18. Joe has three drinking cups: a cylinder-shaped, a cone-shaped and a semi-sphere-shaped. All of them have the same radius and height. The cylinder-shaped cup is full of water. Joe want to pour the water from the cylinder-shaped cup to the other two cups. Which of the following is going to happen?

(a) There will not be enough water to fill the two cups.

(b) The water from the cylinder-shaped cup will exactly fill the two other cups.

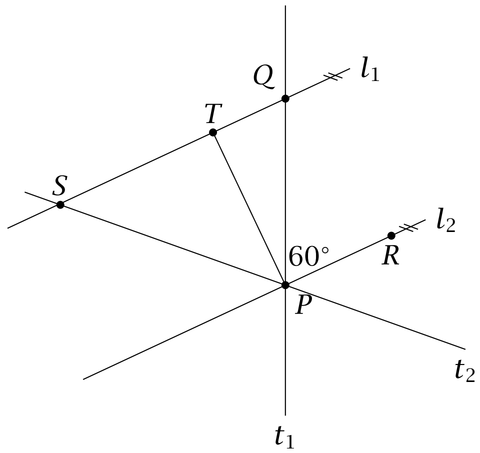
(c) There will be too much water to fit into the other two cups.

(d) It depends on what the radius and height are.

(e) It depends on what the radius is.

SOLUTION **(b)**: For the semi-sphere, the height must be the same as the radius, let's call it r. Since all the cups have the sane radius and height, the radius and height of each of the cup is also r. The volume of a cylinder with radius r and height r is πr^3, the volume of a cone with radius r and height r is $\frac{1}{3}\pi r^3$ and the volume of a semi-sphere with a radius r is $\frac{2}{3}\pi r^3$. So the volume of the semi-sphere and the volume of the cone will exactly add up to the volume of the cylinder.

19. Given that l_1 and l_2 are parallel and t_1 and t_2 and are transversals that intersect l_2 at P. Additionally, $\overline{PT}$ is perpendicular to l_1, $m\angle QPR = 60°$, $PT = TS$ and $PQ = 5$. What is the length of $\overline{PS}$?



(a) $\frac{5\sqrt{3}}{2}$ **(b)** $\frac{\sqrt{6}}{2}$

(c) $\frac{25\sqrt{3}}{2}$ **(d)** $\frac{5\sqrt{6}}{2}$ **(e)** None of the above

SOLUTION **(d)**: Since $\overline{PT}$ is perpendicular to l_1 and l_1 and l_2 are parallel, $\overline{PT}$ is perpendicular to l_2 as well. Therefore $m\angle TPQ = 30°$, and the triangle $\triangle PTQ$ is a $30° - 60° - 90°$ triangle. Therefore $TQ = \frac{5}{2}$ and $TP = \frac{5\sqrt{3}}{2}$. The triangle $\triangle STP$ is a right isosceles triangle with $TS = PT = \frac{5\sqrt{3}}{2}$. Using the Pythagorean Theorem or right triangle trigonometry, we obtain $PS = \frac{5\sqrt{6}}{2}$.

20. David, Bill and George are three thieves. One of them committed a robbery. During the interrogation they made the following statements:

- David: Bill is not the robber. George is the robber.
- Bill: David is innocent. George is the robber.
- George: I am innocent. David is the robber.

It was determined that one of them lied twice, one of them told the truth twice, and one lied once and told the truth once. Who is the robber?

(a) David **(b)** Bill **(c)** George

(d) None of them **(e)** Impossible to determine

Solution **(a)**: Since Bill and George negate each others statements, one of them must be telling the truth twice and one of them must be lying twice. Therefore David must have told the truth once and lied once.
If Bill told the truth twice, then David also told the truth twice, which is a contradiction. Therefore Bill lied twice, and George told the truth twice. David is the robber.

21. Suppose $f(x) = ax + b$. If $f(f(f(x))) = 64x + 63$, what is $a + b$?

(a) 3 **(b)** 7 **(c)** 8 **(d)** 12 **(e)** None of the above

Solution **(b)**: Composition of two linear functions with slopes a_1 and a_2 is a linear function with slope $a_1 a_2$. Therefore $f(f(f(x))) = a^3 x + c$, which gives us $a^3 = 64$, or $a = 4$.
Setting $x = 0$, we get $f(f(f(0))) = 63$, but $f(0) = b$, so the last equality can be written as $f(f(b)) = 63$. But $f(b) = 4b + b = 5b$, which gives us $f(5b) = 63$. Then $f(5b) = 4 \cdot 5b + b = 20b + b = 21b = 63$, which means $b = 3$. Then $a + b = 4 + 3 = 7$.

22. For which values of k does the system

$$x^2 - y^2 = 0$$
$$(x-k)^2 + y^2 = 2$$

have exactly two solutions of the form (x, y) where x and y are real numbers?

(a) $k = \pm 1$ **(b)** $k = \pm 2$ **(c)** $k = 1$ and $k = -2$

(d) $k = -1$ and $k = 2$ **(e)** None of the above

SOLUTION **(b)**: If (x, y) is a solution, then so is $(x, -y)$. The system is equivalent to $y = \pm x$, $(x-k)^2 + x^2 = 2$. The quadratic equation $(x-k)^2 + x^2 = 2$ has discriminant $4k^2 - 8(k^2 - 2) = 4(4 - k^2)$. When the discriminant is 0, we find, for $k = 2$, the solution set is $\{(1, 1), (1, -1)\}$, and when $k = -2$, it is $\{(-1, 1), (-1, -1)\}$. When the discriminant is negative, there are no real solutions x to the quadratic equation, hence no real solutions (x, y) to the system. When the discriminant is positive, there are two distinct real solutions x_1, x_2 to the quadratic equation, at least one of which must be non-zero; hence, the solutions to the system are $(x_1, \pm x_1)$, $(x_2, \pm x_2)$ in this case, and these solutions are at least 3 in number. Therefore, there are exactly 2 values of k for which the system has exactly two distinct solutions, namely, $k = \pm 2$.

23. A library is open every day except Sunday. Max, Gus and Zyk visit the library together for the first time. After the first visit, Max always visits the library two days after her previous visit, except when the library is closed, in which case she goes 3 days after her previous visit. Gus always visits the library three days after his previous visit, except when the library is closed, in which case he goes 4 days after his previous visit. Zyk always makes his next visit 4 days after the previous visit, unless the library is closed, in which case he goes 5 days after his previous visit. If the second time they all visit the library together falls on Friday, what day of the week was their first visit?

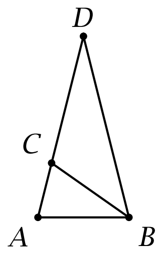
(a) Monday **(b)** Tuesday **(c)** Wednesday **(d)** Thursday **(e)** Friday

SOLUTION **(c)**: Let's make a table showing the patterns of library visits for each person. We denote each day of the week, as customary, with letters M, T, W, R, F and S. We will denote repeating patterns with a bar over the letters, in a similar way repeating decimals are denoted.

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Max	$MWF\overline{MWF}$	$TRS\overline{MWF}$	$WF\overline{MWF}$	$RS\overline{MWF}$	$F\overline{MWF}$	$S\overline{MWF}$
Gus	$MR\overline{MR}$	$TF\overline{MR}$	$WSTF\overline{MR}$	$R\overline{MR}$	$F\overline{MR}$	$STF\overline{MR}$
Zyk	$MFTSW\overline{MFTSW}$	$TSW\overline{MFTSW}$	$W\overline{MFTSW}$	$R\overline{MFTSW}$	$FTSW\overline{MFTSW}$	$SW\overline{MFTSW}$

We see that no matter which day was the first visit, Gus, after a few weeks, settles into the pattern $MRMRMR\ldots$. The only times he will visit the library on Friday is if the first visit is on Tuesday, when his second visit is on Friday of the first week, or if the first visit is on Wednesday, when his fourth visit falls on Friday of the second week, or if the first visit is on Saturday, in which case his third visit is on Friday of the second week. Out of those three cases, however, Max and Zyk will be in the library on the same Friday only if the first visit was on Wednesday.

24. In the triangle shown on the right, $AB = BC = CD$ and $AD = BD$. Find the measure of angle D.



(a) 28° **(b)** 30° **(c)** 36° **(d)** 72° **(e)** None of the above

Solution **(c)**: Let x be the measure of the angle D and y the measure of the angle A. Since the triangle $\triangle BCD$ is isosceles, the measure of angle $\angle CBD$ is also x. Since the triangle $\triangle ABC$ is isosceles, the measure of the angle $\angle BCA$ is also y. Finally, since the triangle $\triangle ADB$ is isosceles, the measure of the angle $\angle ABD$ is also y.

The sum of the angles in the triangle $\triangle ABC$ is $y + y + (y - x) = 3y - x$. The sum of the angles in the triangle $\triangle ADB$ is $2y + x$. This gives us two equations:

$$3y - x = 180$$

$$2y + x = 180$$

Solving this system will give us $x = 36$.

25. Two particles move clockwise around a circle with circumference 300 feet. The faster particle moves at a constant speed of R feet per second, and the slower particle moves at a constant speed of r feet per second. If the particles meet every 50 seconds, then what is the value of $R - r$ in feet per second?

(a) 6 **(b)** 8 **(c)** 10 **(d)** 12 **(e)** 14

Solution **(a)**: The relative speed of the faster particle with respect to the slower particle is $R - r$. The particles will meet every time the faster particle will complete a full circle, measured with respect to the slower particle. Therefore $50(R - r)$ must be 300, and $R - r$ must be 6.