Saginaw Valley State University

2009 Math Olympics – Level II

1. $f(x)$ is a degree three monic polynomial (leading coefficient is 1) such that $f(0) = 3$, $f(1) = 5$ and $f(2) = 11$. What is $f(5)$?

   (a) 27 **(b)** 113 **(c)** 126 **(d)** 173 **(e)** None of the above

2. Which of the following are equal to $\ln \frac{\sqrt{2}+1}{\sqrt{2}-1}$?

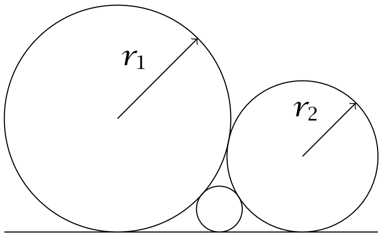
   (a) $2\ln\left(1+\sqrt{2}\right)$ **(b)** $-\ln\left(1-\sqrt{2}\right)^2$ **(c)** $\ln\left(2\sqrt{2}+3\right)$

   (d) All of the above **(e)** None of the above

3. The symbol R_k stands for a positive integer whose base-ten representation is a sequence of k ones, that is $R_1 = 1$, $R_2 = 11$, $R_3 = 111$, etc. The quotient $\frac{R_{24}}{R_4}$ is an integer whose base-ten representation contains only digits 0 and 1. The number of digits 0 in this representation is:

   (a) 6 **(b)** 11 **(c)** 15 **(d)** 16 **(e)** 20

4. Three circles are tangent to each other and to a straight line, as shown in the picture. Express the radius r of the smallest circle in terms of radii r_1 and r_2 of the other two circles.



   (a) $r = \frac{r_1 r_2}{r_1+2\sqrt{r_1 r_2}+r_2}$ **(b)** $r = \frac{r_1 r_2}{r_1-2\sqrt{r_1 r_2}+r_2}$

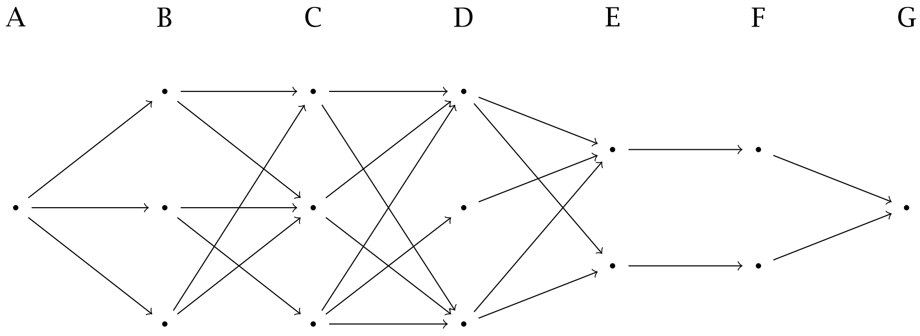
   (c) $r = \frac{r_1 r_2}{r_1+r_2}$ **(d)** $r = r_1 - r_2$

   (e) None of the above

5. A box contains 4 red balls and 6 white balls. A sample of size 3 is drawn without replacement from the box. What is the probability of obtaining 1 red ball and 2 white balls, if you know that at least 2 of the balls in the sample are white?

   (a) 1/2 **(b)** 2/3 **(c)** 3/4 **(d)** 9/11 **(e)** 54/55

6. In the diagram, A, B, ..., G refer to successive states through which a traveler must pass in order to get from A to G, moving from left to right. A path consists of a sequence of line segments leading from one state to the next. A path must always move to the next state until reaching state G. Determine the number of possible paths from A to G.



(a) 20 **(b)** 23 **(c)** 24 **(d)** 25 **(e)** 30

7. In the diagram, the number in the first box is 4 and the number in the second box is 7. Beginning with the second box, the number in each subsequent box is the sum of the numbers in the two boxes immediately adjacent to its box on each side. What is the number in the 2009th box?

4	7	⋯

(a) -3 **(b)** 4 **(c)** -7 **(d)** 7 **(e)** Not enough information given

8. The Awesome Mathematical Machine is capable of two operations. It can increase a number by one, or multiply it by two. The number 0 was entered into the machine. After a while, the machine came up with the result of 2009. What is the smallest number of steps the machine had to perform in order to obtain this result?

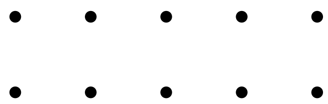
(a) 13 **(b)** 18 **(c)** 21 **(d)** 103 **(e)** None of the above

9. January 1980 had exactly 4 Mondays and 4 Fridays. Which day of the week was January 1st?

(a) Monday **(b)** Tuesday **(c)** Thursday **(d)** Saturday

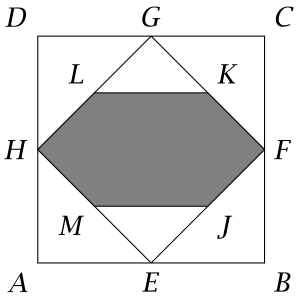
(e) Impossible to determine

10. The points on the right form a square grid. How many isosceles triangles can be drawn with vertices at the points?



(a) 8 **(b)** 16 **(c)** 24 **(d)** 32 **(e)** 40

11. The area of the square $ABCD$ is 64. The midpoints of its sides are joined to form the square $EFGH$. The midpoints of its sides are J, K, L and M. The area of the shaded region is



(a) 23 **(b)** 24 **(c)** 20 **(d)** 28 **(e)** 16

12. Which of the following describes the locus of all points z in the complex plane such that

$$\left|\frac{z-1}{z}\right| = \sqrt{2}?$$

(a) Vertical line through $x = \sqrt{2} - 1$

(b) Circle with center $(-1, 0)$ and radius $\sqrt{2}$

(c) Circle with center $(-1/2, 0)$ and radius $\frac{\sqrt{2}}{2}$

(d) Hyperbola with foci $(\pm 1, 0)$ and eccentricity $\sqrt{2}$

(e) None of the above

13. Let $w = 4 + 3i$ and let $z = \frac{w}{|w|}$. For any positive integer n, let

$$s_n = \sum_{k=1}^{n} z^k = z + z^2 + z^3 + \ldots + z^n$$

Then all s_n lie on a circle with radius:

(a) $\frac{\sqrt{5}}{2}$ **(b)** $\frac{3}{2}$ **(c)** $\frac{\sqrt{10}}{2}$ **(d)** $\frac{10}{3\pi}$ **(e)** They do not lie on a circle

14. How many zeros are there at the end of the product of the first 20 primes?

(a) 0 **(b)** 1 **(c)** 2 **(d)** 5 **(e)** None of the above

15. How many unique solutions of the equation $\tan 2t - 2\cos t = 0$ are in the interval $[0, 2\pi)$?

(a) none **(b)** 2 **(c)** 4 **(d)** 5

(e) infinitely many

16. Find the sum of the first 100 common terms in the sequences
17, 21, 25, 29, ...
16, 21, 26, 31, ...

(a) 21500 **(b)** 26350 **(c)** 47850 **(d)** 101100 **(e)** None of the above

17. Simplify

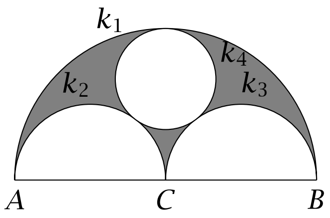
$$\sqrt{13+30\sqrt{2+\sqrt{9+4\sqrt{2}}}}-\sqrt{18}.$$

(a) 0 **(b)** 3 **(c)** 5 **(d)** $3\sqrt{2}$ **(e)** None of the above

18. How many prime numbers less than ten thousand have digits that add up to 2?

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** 5

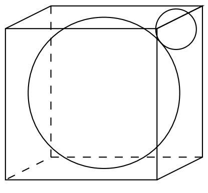
19. In the drawing on the right, $AB = 2R$, C is the midpoint of AB, k_1, k_2 and k_3 are semicircles with diameters AB, AC and CB, respectively. The circle k_4 is tangent to k_1, k_2 and k_3. Find the shaded area in terms of R.



(a) $\frac{\pi R^2}{4}$ **(b)** $\frac{3\pi R^2}{16}$ **(c)** $\frac{5\pi R^2}{36}$ **(d)** $\frac{7\pi R^2}{32}$

(e) None of the above

20. A sphere with a radius R is inscribed in a cube. Another (smaller) sphere is placed inside a corner of the cube so that it is tangent to three sides of the cube and to the big sphere (see the simplified picture on the right). Find the radius of the small sphere in terms of R.



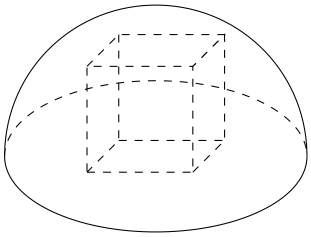
(a) $r = \frac{(\sqrt{3}-1)R}{2}$ **(b)** $r = (2-\sqrt{3})\,R$ **(c)** $r = \frac{(\sqrt{2}-1)R}{2}$

(d) $r = \left(2-\sqrt{2}\right)R$ **(e)** None of the above

21. Find $\tan 9^\circ - \tan 27^\circ - \tan 63^\circ + \tan 81^\circ$.

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** None of the above

22. A cube is inscribed in a half sphere so that four of its vertices are on the base and the other four are on the sphere. Find the ratio of the volume of the cube to the volume of the half ball.



(a) $\frac{\sqrt{3}}{\pi\sqrt{2}}$

(b) $\frac{\sqrt{2}}{\pi\sqrt{3}}$ **(c)** $\frac{\sqrt{3}}{\pi}$ **(d)** $\frac{\sqrt{2}}{\pi}$ **(e)** None of the above

23. If $s = \cos^2\alpha + \cos^2\beta$, find $\cos(\alpha+\beta)\cos(\alpha-\beta)$ in terms of s.

(a) $s - 1$ **(b)** $\frac{s-1}{s}$ **(c)** s^2 **(d)** $\sqrt{2}$ **(e)** None of the above

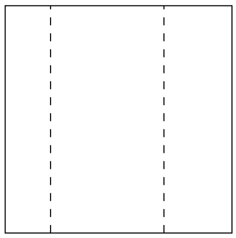
24. In the expression

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f}$$

each letter is replaced by one of the digits 1, 2, 3, 4, 5 and 6. What is the largest possible value of an expression that can be obtained in this way.

(a) $8\frac{2}{3}$ **(b)** $9\frac{5}{6}$ **(c)** $9\frac{1}{3}$ **(d)** $9\frac{2}{3}$ **(e)** $10\frac{1}{3}$

25. A square is cut into three rectangles along two lines parallel to a side, as shown. If the perimeters of of the three rectangles are 22, 26 and 24, respectively, then the area of the original square is



(a) 24 **(b)** 36 **(c)** 64 **(d)** 81 **(e)** 96