Saginaw Valley State University

2009 Math Olympics – Level I Solutions

1. A man and his wife take a trip that usually takes three hours if they drive at an average speed of 60 mi/h. After an hour and a half of driving at a steady rate of 60 mi/h they stop for a 45 minute lunch break. How fast do they need to drive on the rest of the trip to arrive three hours after they started?

   (a) 90 mi/h **(b)** 120 mi/h **(c)** 80 mi/h **(d)** 105 mi/h **(e)** None of the above

   SOLUTION **(b)**: The distance of the trip is 3 h × 60 mi/h = 180 mi. Before the stop they have already driven 1.5 h × 60 mi/h = 90 mi (the half of the trip). After the lunch break they have 45 minutes left to cover the remaining 90 mi. The have to drive at the speed

$$\frac{90\text{ mi}}{\frac{3}{4}\text{ h}} = 90\cdot\frac{4}{3}\text{ mi/h} = 120\text{ mi/h}.$$

2. Ali and Fred are driving to Bing's house. Ali confuses left with right 50% of the time (This means half the time, when he means to say "left" he says "right" instead, and half the time when he means to say "left" he really says "left".). Fred gets left and right confused 25% of the time. (This means when someone tells him to turn left, 75% of the time he will turn left, and 25% he will turn right instead, and when someone tells him to turn right, 75% of the time he will turn right, and 25% he will turn left instead.) If Fred is driving and Ali is giving directions, and there are two left turns on the way to Bing's house (and no other turns) what is the probability they will get there?

   (a) $\frac{1}{4}$ **(b)** $\frac{9}{64}$ **(c)** $\frac{1}{2}$ **(d)** $\frac{3}{4}$ **(e)** None of the above

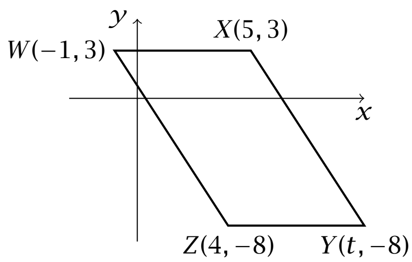
   SOLUTION **(a)**: At each left turn, the probability of actually turning left is equal to the probability of Ali actually saying left times the probability of Fred turning the way he is told, plus the probability of Ali accidentally directing Fred to turn right times the probability that Fred turns left while told to turn right. In numbers:

$$P(\text{turning left}) = \frac{1}{2}\cdot\frac{3}{4}+\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{2}$$

   Since they are two left turns, and they are independent of each other, the probability of arriving to Bing's house is

$$\frac{1}{2}\frac{1}{2}=\frac{1}{4}.$$

3. If $WXYZ$ is a parallelogram, the t equals:



   (a) 8 **(b)** 9 **(c)** 10 **(d)** 11 **(e)** 12

   SOLUTION **(c)**: In a parallelogram, the lengths of parallel sides are equal. Therefore the length of the horizontal side ZY is the same as the length of the horizontal side WX, which is 6. Since the side ZY is horizontal (the y-coordinates of the vertices are both -8), the x-coordinate of Y is the x-coordinate of Z increased by 6. Therefore $t = 4 + 6 = 10$.

4. The symbol R_k stands for a positive integer whose base-ten representation is a sequence of k ones, that is $R_1 = 1$, $R_2 = 11$, $R_3 = 111$, etc. The quotient $\frac{R_{24}}{R_4}$ is an integer whose base-ten representation contains only digits 0 and 1. The number of digits 0 in this representation is:

   (a) 6 **(b)** 13 **(c)** 15 **(d)** 16 **(e)** 20

   SOLUTION **(c)**: Using a decimal expansion, $R_k = 1 + 10 + 10^2 + 10^3 + \ldots + 10^{k-1}$. This is a finite geometric series with common ratio 10 and first term 1. Therefore

$$R_k = \frac{1 - 10^k}{1 - 10} = \frac{10^k - 1}{9}$$

   and

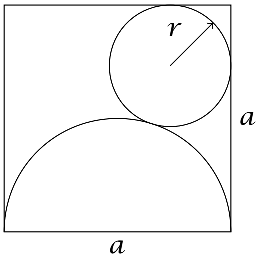
$$\frac{R_{24}}{R_4} = \frac{\frac{10^{24}-1}{9}}{\frac{10^4-1}{9}} = \frac{10^{24} - 1}{10^4 - 1} = \frac{\left(10^4\right)^6 - 1}{10^4 - 1}$$

   The last is the sum of the first six terms of the geometric series with first term 1 and common ratio 10^4, which means

$$\frac{R_{24}}{R_4} = 1 + 10^4 + 10^8 + 10^{12} + 10^{16} + 10^{20} = 100,010,001,000,100,010,001$$

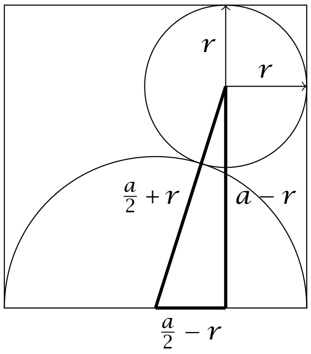
   which has 15 zeros.

5. A square with side a has a semicircle constructed inside it such that the diameter of the semicircle is one of the sides of the square. A circle with maximal radius is then constructed inside the square, but outside of the semicircle (see illustration). Express the radius r of the circle in terms of a.



(a) $r = a(2 - \sqrt{3})$ **(b)** $r = a/4$

(c) $r = a(3 - \sqrt{2})$ **(d)** $r = \sqrt{a}/2$ **(e)** None of the above



SOLUTION **(a):** The largest circle that is outside the semicircle and inside the square has to be tangent to both sides of the square and to the semicircle. In the picture on the left, the triangle with sides $\frac{a}{2} + r$, $\frac{a}{2} - r$ and $a - r$ is a right triangle, with hypotenuse $\frac{a}{2} + r$. Therefore

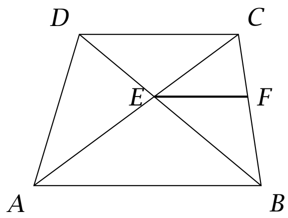
$$\begin{aligned}
\left(\frac{a}{2} + r\right)^2 &= \left(\frac{a}{2} - r\right)^2 + (a - r)^2 \\
\frac{a^2}{4} + ar + r^2 &= \frac{a^2}{4} - ar + r^2 + a^2 - 2ar + r^2 \\
ar &= -3ar + a^2 + r^2 \\
r^2 - 4ar + a^2 &= 0
\end{aligned}$$

This is a quadratic equation in r, and can be solved, for example, by completing the square.

$$\begin{aligned}
r^2 - 4ar + 4a^2 &= 3a^2 \\
(r - 2a)^2 &= 3a^2 \\
r - 2a &= \pm\sqrt{3}a
\end{aligned}$$

That gives us $r = a\,(2 \pm \sqrt{3})$, but since the circle is inside the square, $r < a$, and the solution is $a\,(2 - \sqrt{3})$.

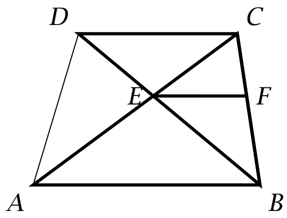
6. In the trapezoid $ABCD$ (the picture is not drawn to scale), the sides AB and CD are parallel, and $|AB| = 4$ and $|CD| = 3$. The point E is the intersection point of the two diagonals, and the segment EF is parallel to AB and CD. The length of EF is



(a) 3 **(b)** 6

(c) 12/7 **(d)** 3/2

(e) Not enough information given



Solution **(c)**: The problem can be solved using two pairs of similar triangles: $\triangle CDB$ and $\triangle FEB$, and $\triangle ABC$ and $\triangle EFC$. These pairs of similar triangles will give us proportions:

$$\frac{|EF|}{|AB|} = \frac{|CF|}{|BC|}$$

and

$$\frac{|EF|}{|CD|} = \frac{|BF|}{|BC|}$$

Adding these two proportions together will give us

$$\frac{|EF|}{|AB|} + \frac{|EF|}{|CD|} = \frac{|BF|}{|BC|} + \frac{|FC|}{|BC|} = \frac{|BF| + |FC|}{|BC|} = \frac{|BC|}{|BC|} = 1$$

Plugging in the known values for $|AB|$ and $|CD|$ gives us

$$\frac{|EF|}{4} + \frac{|EF|}{3} = 1$$

or

$$\frac{3|EF|}{12} + \frac{4|EF|}{12} = \frac{7|EF|}{12} = 1$$

or

$$|EF| = \frac{12}{7}$$

7. A company offers three types of benefits, A, B, and C. As part of this plan, the individual employees may choose either exactly two benefit types, or no benefits at all. The proportion of the company's employees that choose benefit types A, B and C are $1/4$, $1/3$ and $5/12$, respectively. Determine the probability that a randomly chosen employee will choose no benefits at all.

(a) 0 **(b)** $47/144$ **(c)** $1/2$ **(d)** $97/144$ **(e)** $7/9$

SOLUTION **(c)**: Let x be the proportion of employees that chose benefits A and B, y the proportion of employees who chose the benefits A and C, and z the proportion of employees who chose the benefits B and C. Since they may only choose either exactly two benefits or no benefit at all, $1-(x+y+z)$ must be the proportion of employees who chose no benefit at all.

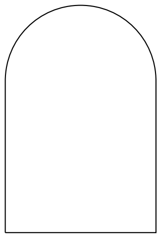
Since the employees who chose benefit A are exactly those who chose either benefits A and B or benefits A and C, $x+y=1/4$. Similarly, $x+z=1/3$ and $y+z=5/12$. Adding these three equations together gives us

$$2x+2y+2z=\frac{1}{4}+\frac{1}{3}+\frac{5}{12}=1$$

so

$$x+y+z=\frac{1}{2}.$$

Therefore the proportion of employees who chose no benefits at all is $1-1/2=1/2$.



8. The (possibly out-of-scale) picture shows a window whose shape consists of a semicircle on top of a square. If the perimeter of the window is 20 ft, what is the exact length of the bottom side of the window?

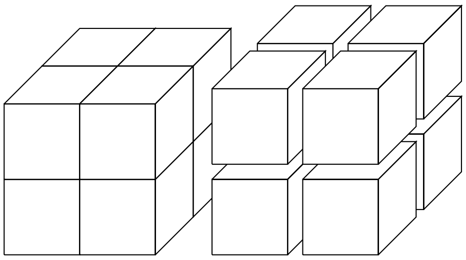
(a) $\frac{40}{\pi+3}$ ft **(b)** $\frac{40}{\pi+6}$ ft **(c)** $\frac{20}{2\pi+3}$ ft **(d)** $\frac{20}{\pi+3}$ ft **(e)** None of the above

SOLUTION **(b)**: If the length of the bottom side of the window is x, then the perimeter will be $3x + \frac{\pi x}{2}$. This will give us the equation

$$(3 + \frac{\pi}{2})x = 20.$$

Solving for x will give us

$$x = \frac{20}{3 + \frac{\pi}{2}} = \frac{40}{6 + \pi}.$$

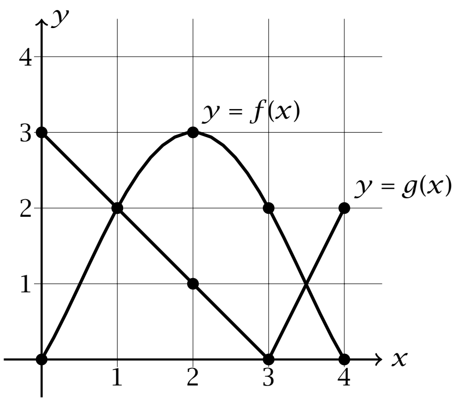


9. A rectangular parallelepiped is cut into 8 smaller rectangular parallelepipeds by three cuts parallel to the sides. The 8 smaller solids are then separated, as shown in the picture. What is the percent increase in the total surface area?

(a) 25% **(b)** 33% **(c)** 50% **(d)** 75% **(e)** 100%

SOLUTION **(e)**: A rectangular parallelepiped has 6 rectangular sides. The top and bottom sides are congruent, as are the front and back sides, as well as the left and right sides. When making a cut parallel to two of the sides, we create two new rectangular surfaces, congruent to the given sides. Therefore the number of each congruent rectangles will double, and so will the total surface area.
Another way: After cutting, but before separating the small solids, the parallelepiped consists of 8 smaller parallelepipeds. Each of those has exactly three sides, that is half of its total surface, exposed, while the other half of its surface remains hidden. Separating the solids exposes these hidden surfaces, therefore doubling the total surface area.

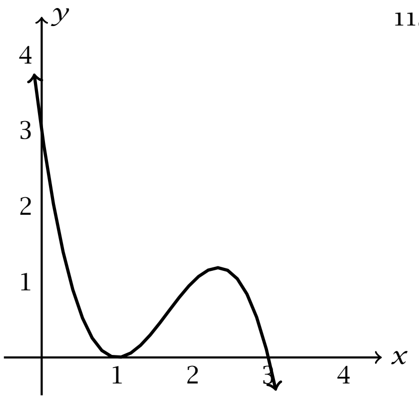
10. Find $(g \circ f)(1)$.



(a) 1

(b) 2 **(c)** 3 **(d)** 4 **(e)** Not enough information given

SOLUTION **(a)**: $(g \circ f)(1) = g(f(1)) = g(2) = 1$



11. The picture shows the graph of a third degree polynomial $f(x)$. Which of the following gives $f(x)$?

(a) $f(x) = (x-1)(x-3)^2$

(b) $f(x) = (x-1)^2(x-3)$ **(c)** $f(x) = (1-x)^2(3-x)$

(d) $f(x) = (1-x)(3-x)^2$ **(e)** $f(x) = (x^2+1)(x-3)$

Solution **(c)**: The polynomial has a simple zero at 3 and a zero of an even multiplicity at 1. Since the degree is 3, the sum of multiplicities of all of its zeros must be 3, which means that the zero at 1 is of multiplicity 2 and that there are no other zeros. The polynomial must then be of the form

$$f(x) = a(x-1)^2(x-3).$$

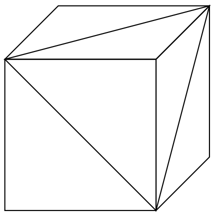
The y-intercept is at $(0,3)$, which means that $f(0) = 3$, or

$$f(0) = a(0-1)^2(0-3) = a(-3) = 3$$

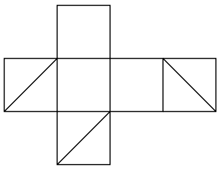
which means that $a = -1$. The polynomial is then

$$f(x) = -(x-1)^2(x-3) = (1-x)^2(3-x).$$

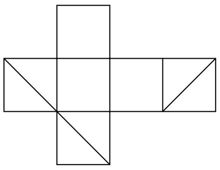
12. On three sides of a cube, diagonals are drawn as shown in the picture. If we unfold the cube, which of the following could we obtain?



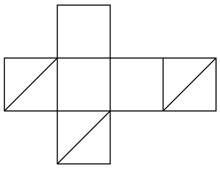
(a)



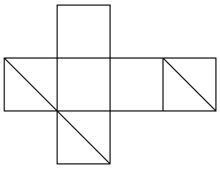
(b)



(c)



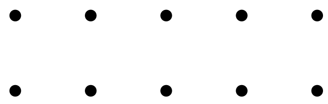
(d)



(e) None of the above

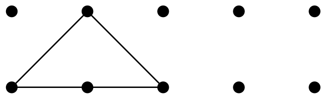
Solution **(b)**: Options **(a)** and **(c)** cannot be obtained by unfolding the cube, since the diagonals in the leftmost square and the lowest square do not connect. In **(d)**, the diagonals in those two square connect, but the diagonal in the rightmost square does not connect to either of them. The only option that can be obtained by unfolding the cube is **(b)**.

13. The ten points on the right form a square grid. How many isosceles triangles can be drawn with vertices at the points?

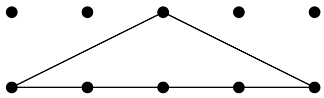


(a) 8 **(b)** 16 **(c)** 24 **(d)** 32 **(e)** 40

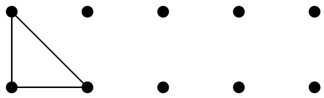
SOLUTION **(c)**: There will be 3 triangles of this type:



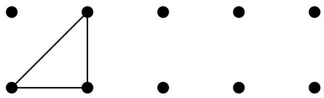
and one triangle like this:



Another 4 triangles can be obtained from the above by turning them upside down.
There will be 4 triangles like this one:



and 4 like this:



and 8 more like the 8 previous ones turned upside down.

14. The numbers from 1 to 100 are written onto slips of paper and placed into a hat. How many numbers do we have to pull out of the hat to make sure that their product is divisible by 10?

(a) 52 **(b)** 80 **(c)** 81 **(d)** 82 **(e)** 91

SOLUTION **(c)**: In order for the product to be divisible by 10, at least one of the numbers must be even, and one must be a multiple of 5. There are 50 even and 50 odd numbers in the hat, so to make sure we have at least one even number, we need to pull out at least 51 numbers. There are 20 multiples of 5 and 80 numbers relatively prime to 5, so to make sure there is at least one multiple of 5, we need to pull out at least 81 numbers. By then, we will have at least one[1] even numbers, so the product will be divisible by 10.

[1] In fact, at least 30

15. Simplify

$$\sqrt{\frac{4+\sqrt{15}}{4-\sqrt{15}}}+\sqrt{\frac{4-\sqrt{15}}{4+\sqrt{15}}}$$

(a) 0 **(b)** 2 **(c)** 8 **(d)** 19 **(e)** None of the above

SOLUTION **(c)**:

$$\sqrt{\frac{4+\sqrt{15}}{4-\sqrt{15}}}+\sqrt{\frac{4-\sqrt{15}}{4+\sqrt{15}}}=\frac{\sqrt{4+\sqrt{15}}}{\sqrt{4-\sqrt{15}}}+\frac{\sqrt{4-\sqrt{15}}}{\sqrt{4+\sqrt{15}}}$$

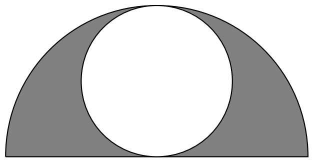
Combine the fractions: $$=\frac{\sqrt{4+\sqrt{15}}^{\,2}+\sqrt{4-\sqrt{15}}^{\,2}}{\sqrt{4-\sqrt{15}}\sqrt{4+\sqrt{15}}}$$

Simplify the numerator: $$=\frac{4+\sqrt{15}+4-\sqrt{15}}{\sqrt{4-\sqrt{15}}\sqrt{4+\sqrt{15}}}$$

Use difference of squares: $$=\frac{4+\sqrt{15}+4-\sqrt{15}}{\sqrt{16-15}}$$

$$=\frac{8}{1}=8$$

16. In the drawing on the right, the small white circle is the largest circle that fits inside the semicircle. What is the ratio of the area of the small circle to the shaded area?



(a) $2:3$ **(b)** $1:1$ **(c)** $1:2$ **(d)** $3:4$ **(e)** $2:\pi$

SOLUTION **(b)**: The radius of the small circle is exactly half the radius of the semicircle. If we call the radius of the small circle r, then the area of the small circle is πr^2 and the area of the semicircle is $\frac{1}{2}\pi(2r)^2 = 2\pi r^2$. The shaded area is then $2\pi r^2 - \pi r^2 = \pi r^2$, which is the same as the area of the small circle. The ratio is $1:1$.

17. What is 5 % of 25 % of 4000?

(a) 32 **(b)** 50 **(c)** 125 **(d)** 200 **(e)** 500

SOLUTION **(b)**: The answer is $.05\cdot(.25\cdot 4000) = .05\cdot 1000 = 50$.

18. If $a * b$ is the greatest common factor of two positive integers a and b, how many of the following equations are *not* always true?

$$a * 1 = a \qquad a * b = b * a$$
$$a * (b * c) = (a * b) * c \qquad a * (b + c) = a * b + a * c$$

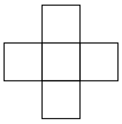
(a) 0 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** 4

Solution **(c)**: There are two equations that are not always true:

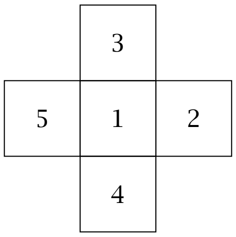
– $a * 1 = a$: The greatest common factor of a and 1 is 1, so this is true only if $a = 1$.

– $a * (b + c) = a * b + a * c$: for example $5 * (2 + 3) = 5 * 5 = 5$ but $5 * 2 + 5 * 3 = 1 + 1 = 2$.
The other two equations are always true:
By definition, the greatest common factor does not depend on the order. Also, both $a * (b * c)$ and $(a * b) * c$ refer to the greatest common factor of the three integers, a, b and c.

19. Suppose we arrange the numbers 1, 2, 3, 4 and 5 in the five squares so that the horizontal and the vertical line both add up to 8. Which number has to go in the middle square?



(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** 5



Solution **(a)**: If add the sum of all three numbers on the horizontal line to the sum of all three numbers on the vertical line, we will get $8 + 8 = 16$. In this sum, each of the numbers appears exactly once except for the one in the middle square, which appears twice. If we call this number a, then

$$1 + 2 + 3 + 4 + 5 + a = 16$$

or $15 + a = 16$, so a must be 1.

20. For integers a, b and c, define $(a\uparrow b\downarrow c)$ to mean $a^b - b^c + c^a$. Then $(1\uparrow -1\downarrow 2)$ equals

(a) -4 **(b)** -2 **(c)** 0 **(d)** 2 **(e)** 4

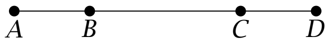
Solution **(d)**: $(1\uparrow -1\downarrow 2) = 1^{-1} - (-1)^2 + 2^1 = 1 - 1 + 2 = 2$.

21. A frog starts out by hopping one inch. Each successive hop is 2 inches longer than the previous hop. How far has he gone all together at the end of his 13^{th} hop?

(a) 25 in **(b)** 169 in **(c)** 91 in **(d)** 157 in **(e)** None of the above

Solution **(b)**: This is the sum of an arithmetic sequence. The formula for the i-th term term in the sequence is $a_i = 1 + (i - 1)2 = 2i - 1$. The formula for the n^{th} partial sum can be derived (if not known) by $\sum_{i=1}^{n} a_i = 2\sum_{i=1}^{n} i - \sum_{i=1}^{n-1} 1 = 2\frac{(n-1)n}{2} - n = n^2 + n - n = n^2$. So after the 13^{th} jump the frog has gone 169 inches.

22. Four points are on a line segment, as shown. If $|AB| : |BC| = 1 : 2$ and $|BC| : |CD| = 8 : 5$, then $|AB| : |BD|$ equals



(a) $4 : 13$ **(b)** $1 : 13$ **(c)** $1 : 7$ **(d)** $3 : 13$ **(e)** $4 : 17$

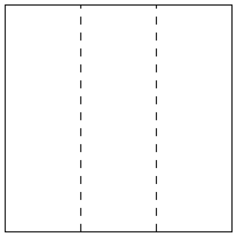
Solution **(a)**: Since $|BD| = |BC| + |CD|$, we have

$$\frac{|BD|}{|CD|} = \frac{|BC|}{|CD|} + \frac{|CD|}{|CD|} = \frac{8}{5} + \frac{5}{5} = \frac{13}{5}.$$

Then

$$\frac{|AB|}{|BD|} = \frac{|AB|}{|BC|} \cdot \frac{|BC|}{|CD|} \cdot \frac{|CD|}{|BD|} = \frac{1}{2} \cdot \frac{8}{5} \cdot \frac{5}{13} = \frac{4}{13}.$$

23. A square is cut into three congruent rectangles along two lines parallel to a side, as shown. If the perimeter of each of the three rectangles is 24, then the area of the original square is



(a) 24 **(b)** 36 **(c)** 64 **(d)** 81 **(e)** 96

Solution **(d)**: Let $3a$ be the side of the square. Then each of the rectangles will have dimensions $3a$ and a, resulting in the perimeter of $6a + 2a = 8a$. As the perimeter is 24, a must be 3, the side of the square must have length 9, and its area is 81.

24. Which of the following triangles cannot exist?

(a) An acute isosceles triangle **(b)** An isosceles right triangle **(c)** An obtuse right triangle

(d) A scalene right triangle **(e)** A scalene obtuse triangle

Solution **(c)**: An obtuse right triangle would have to have a right angle and an obtuse angle, causing the sum of all angles in the triangle to exceed $180°$, which is not possible.

25. Friday the 13^{th} occurred two months in a row this year. What is the next year that this will happen?

(a) 2021 **(b)** 2016 **(c)** 2015 **(d)** 2014 **(e)** None of the above

Solution **(c)**: Friday the 13^{th} can happen two months in a row only when it happens the first time in February, and it is not a leap year. So the next time it can happen again is when February begins on the same day that it did this year (Sunday), and it is not a leap year. As there are 365 days in a regular year, 1 more day than 52 weeks, and 366 days in a leap year, for each regular year, the month starts one day later than it did the year before, and for each leap year it will start two days later. For example next year February begins on Monday, but if this were a leap year, it would begin on Tuesday. Leap years occur every 4 years, in years that are multiples of 4. Since there is only one leap year in the next 7 years (2012) the next time February begins on a Sunday is in 6 years in 2015.