Saginaw Valley State University

2007 Math Olympics – Level II

1. For which x is $\log_2(x^2 - 2x - 2) \le 0$?

   (a) $(-1, 3)$ **(b)** $[-1, 3]$

   (c) $(-\infty, 1-\sqrt{3}) \cup (1+\sqrt{3}, \infty)$ **(d)** $[-1, 1-\sqrt{3}) \cup (1+\sqrt{3}, 3]$

   (e) None of the above

2. The exact value of the expression $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is

   (a) $\frac{2\pi}{3}$ **(b)** $\frac{\sqrt{3}}{2}$ **(c)** $\frac{\pi}{3}$ **(d)** $\sqrt{3}$ **(e)** None of the above

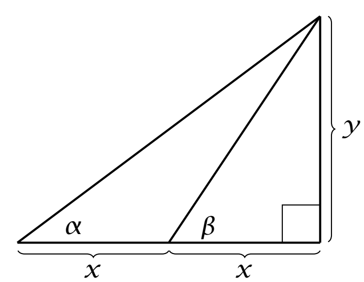
3. Augustus, Benedict, Claudio, and Diana have been accused of stealing the golden mean. It is known that one of these four people must have done it. Augustus says “Benedict did it”. Benedict says “Diana did it”. Claudio says “I didn’t do it”. Diana says “Benedict is lying when he says I did it”. If it is known that exactly one of them is lying, which one did it?

   (a) Augustus **(b)** Benedict

   (c) Claudio **(d)** Diana

   (e) More than one person must be lying

4. Which of the following equations expresses the correct relationship between the angles α and β as defined in the figure?



   (a) $\beta = \frac{1}{2}\sin^{-1}\left(\frac{1-5\sin^2(\alpha)}{1+3\sin^2(\alpha)}\right)$

   (b) $\beta = \frac{1}{2}\cos^{-1}\left(\frac{1-5\cos^2(\alpha)}{1+3\cos^2(\alpha)}\right)$ **(c)** $\beta = \frac{1}{2}\cos^{-1}\left(\frac{1-5\sin^2(\alpha)}{1+3\sin^2(\alpha)}\right)$

   (d) $\beta = \frac{1}{2}\sin^{-1}\left(\frac{1-5\cos^2(\alpha)}{1+3\cos^2(\alpha)}\right)$ **(e)** None of the above

5. The number of solutions to the equation $2^x + 4(2^{-x}) = 5$ is

   (a) 0 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** More than 3

6. The expression $\frac{1+\cos\theta}{\sin\theta} + \frac{\sin\theta}{1+\cos\theta}$ is equivalent to which of the following? (Assume $\theta \ne \pi + 2\pi n$ for any integer n.)

   (a) $2\csc\theta$ **(b)** $2\cot\theta$ **(c)** $2\sec\theta$ **(d)** $\csc\theta\sec\theta$

   (e) None of the above

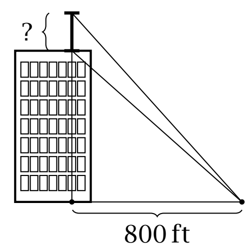
7. 10% of a high school senior class participate in Math Olympics. 95% of the seniors that participate in Math Olympics get into the college of their choice. Only 50% of the seniors who don't participate in Math Olympics get into the college of their choice. If a senior from that high school got into the college of his or her choice, what is the probability that this senior participated in Math Olympics?

(a) .095 **(b)** .545 **(c)** $\frac{19}{109}$ **(d)** .95 **(e)** None of the above

8. Let F be a function such that $F\left(\frac{8}{\sqrt{1+\sqrt{x}}}\right) = x$ for all $x \geq 0$. What is $F(4)$?

(a) 9 **(b)** $\frac{8\sqrt{3}}{3}$ **(c)** $\frac{-12+8\sqrt{3}}{3}$ **(d)** $\frac{9}{16}$ **(e)** None of the above

9. The largest advertising sign in the world is a large capital "I" on top of a building in L.A. At a distance of 800 ft along the ground from a point directly below the sign, the angle of elevation to the top of the sign is 45°. From this same point, the angle of elevation to the bottom of the sign is 30° . What is the height of the sign?



(a) 800 ft **(b)** $\frac{800\sqrt{3}}{3}$ ft

(c) $\frac{800(3-\sqrt{3})}{3}$ ft **(d)** $800\left(\sqrt{3}-1\right)$ ft **(e)** None of the above

10. A middle school has 100 lockers numbered 1 to 100, and 100 students. The first student goes down the row of lockers and opens every locker. Then the second student goes down the row of lockers and closes every locker that is numbered with a multiple of two. Then the third student goes down the row of lockers, and for every locker that is numbered with a multiple of 3, if it is open, she closes it, but if it is already closed, she opens it again. The fourth student then does the same thing for the lockers numbered with multiple of 4, and so on, down to the hundreth student. In the end, how many lockers are still open?

(a) 1 **(b)** all of the lockers that are not numbered with prime numbers **(c)** 10

(d) 15 **(e)** None of the above

11. Which of the following is the set of all real solutions to the equation $2^{3x} - 5\left(2^{2x}\right) + 2^{x+2} + 6 = 0$?

(a) $\left\{\frac{\ln(3)}{\ln(2)}, \frac{\ln(2-\sqrt{3})}{\ln(2)}\right\}$ **(b)** $\left\{\frac{\ln(3)}{\ln(2)}, \frac{\ln(1+\sqrt{3})}{\ln(2)}\right\}$ **(c)** $\left\{\frac{\ln(1+\sqrt{3})}{\ln(2)}, \frac{\ln(1+\sqrt{5})}{\ln(2)}\right\}$

(d) $\left\{\frac{\ln(3)}{\ln(2)}, \frac{\ln(1-\sqrt{3})}{\ln(2)}, \frac{\ln(1+\sqrt{3})}{\ln(2)}\right\}$ **(e)** None of the above

12. The line $3x + y = b$ and the equation $2x^2 + y^2 = 1$ are graphed on an (x, y)-rectangular coordinate system. For what values of b will the graph of the line $3x + y = b$ be tangent to the graph of the equation $2x^2 + y^2 = 1$?

    (a) The graph of the line $3x + y = b$ is not tangent to the graph of $2x^2 + y^2 = 1$ for any b.

    (b) The graph of the line $3x + y = b$ is tangent to the graph of $2x^2 + y^2 = 1$ for all b.

    (c) $b = \pm 1$ **(d)** $b = \pm\frac{\sqrt{22}}{2}$ **(e)** None of the above

13. Consider all three digit numbers for which the tens digit is the sum of the ones digit and the hundreds digit. How many such numbers are divisible by 11? (Note that 0 cannot be the first digit of a three digit number.)

    (a) 9 **(b)** 10 **(c)** 45 **(d)** 55 **(e)** None of the above

14. Simplify

$$\left(\frac{1}{\sqrt{a}+\sqrt{a+1}} + \frac{1}{\sqrt{a}-\sqrt{a-1}}\right) \div \left(1 + \sqrt{\frac{a+1}{a-1}}\right) \quad \text{where } a > 1$$

    (a) $\sqrt{a} + \sqrt{a+1}$ **(b)** $\frac{1}{\sqrt{a}-\sqrt{a-1}}$ **(c)** $\sqrt{a-1}$ **(d)** $\frac{1}{\sqrt{a+1}}$

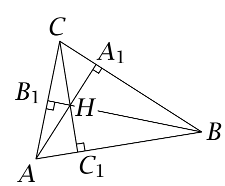
    (e) None of the above

15. Find the sum of all irreducible fractions with a denominator 3 that are between the positive integers m and n, where $m < n$.

    (a) $\frac{m(n-m)}{3}$ **(b)** $n^2 - m^2$ **(c)** $\frac{n^2-m^2}{3}$ **(d)** $n^2 - m^2 - 2m$

    (e) None of the above

16. The heights (altitudes) AA_1, BB_1, CC_1 in the triangle ABC intersect at the point H so that $CH = HC_1$. If $\alpha = \angle CAB$ and $\beta = \angle CBA$ then $\tan\alpha \tan\beta$ is



    (a) $\frac{1}{2}$ **(b)** 2 **(c)** $\frac{1}{4}$ **(d)** 4 **(e)** None of the above

17. A license plate code consists of two letters followed by three digits. The letters cannot repeat, but the numbers can. What is the probability that a randomly chosen plate has at least one zero?

    (a) $\frac{3}{10}$ **(b)** $\frac{1}{10}$ **(c)** $\frac{729}{1000}$ **(d)** $\frac{271}{1000}$ **(e)** None of the above

18. How many integers between 1 and 1000 inclusive are neither multiples of 4 nor multiples of 7?

    (a) 357 **(b)** 663 **(c)** 608 **(d)** 643 **(e)** None of the above

19. Suppose that $f(x) = 2\,(f(x+1) + f(x-1))$ for all x. If $f(2) = 2$ and $f(4) = -2$, what is $f(7)$?

(a) -1 **(b)** $\frac{3}{2}$ **(c)** $\frac{7}{4}$ **(d)** $-\frac{5}{8}$ **(e)** None of the above

20. Suppose $p(x)$ is a degree 4 polynomial with rational coefficients. If 1 is the only real root of $p(x)$, but $p(x)$ is not a multiple of $(x-1)^4$, which of the following must also be true?

(a) 1 must be a root of multiplicity one.

(b) $p(x)$ has two roots of multiplicity two.

(c) 1 is the only root of $p(x)$ of multiplicity two.

(d) 1 must be a root of multiplicity three.

(e) Not enough information is given.

21. Find the value of $\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right) + \tan\left(\cos^{-1}\left(\frac{5}{7}\right)\right)$.

(a) $\frac{11}{5}$ **(b)** $\frac{4+2\sqrt{6}}{5}$ **(c)** $\frac{\pi}{6}$ **(d)** $\frac{8\sqrt{6}+25}{10\sqrt{6}}$ **(e)** None of the above

22. The equation $\log x^2 = (\log x)^2$ has

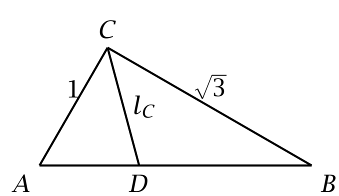
(a) no solution **(b)** one solution **(c)** two solutions

(d) three solutions **(e)** more than three solutions

23. If $a^y = x$, what is $\log_x \frac{1}{a}$? (Assume $a > 0$, $x > 0$ and $x \neq 1$.)

(a) $\frac{1}{y}$ **(b)** $-y$ **(c)** $-a$ **(d)** $-\frac{1}{y}$ **(e)** None of the above

24. A triangle ABC has sides $\overline{AB}$ of length 2 ft, $\overline{AC}$ of length 1 ft and $\overline{BC}$ of length $\sqrt{3}$ ft. The angle bisector l_C of $\angle ACB$ intersects the side AB at the point D. Find AD.



(a) $\sqrt{3} - 1$

(b) $\sqrt{3} + 1$ **(c)** $\frac{2}{3}$ **(d)** $\frac{2}{\sqrt{3}}$ **(e)** None of the above

25. A couple plans to have 6 children. Assuming that boys and girls are equally likely to be born, what is the probability that they will have exactly 3 of each?

(a) $\frac{5}{16}$ **(b)** .5 **(c)** $\frac{1}{32}$ **(d)** $\frac{1}{8}$ **(e)** None of the above