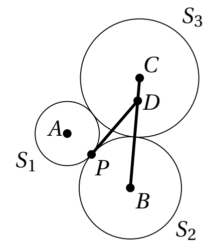
Saginaw Valley State University
2006 Math Olympics – Level II

1. How many zeros are at the end of 79!
   (Recall, $n!$ is the product $1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.)

   (a) none **(b)** 18 **(c)** 15 **(d)** 7 **(e)** None of the above

2. Three circles, S_1, S_2 and S_3, of different radii with centers A, B, C, respectively, are drawn so that any two touch each other. The point P is the intersection of the smaller circles S_1 and S_2. The common tangent to S_1 and S_2 is extended from P and meets BC at D. If the radii of S_1, S_2 and S_3 are $2, 3$, and 10, respectively, find the length of CD.



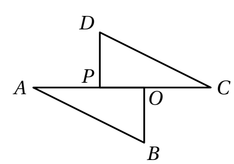
   (a) 6 **(b)** $\frac{26}{5}$ **(c)** $\frac{13}{3}$ **(d)** $\frac{13}{5}$ **(e)** None of the above

3. Let $n = \sqrt{6+\sqrt{11}} + \sqrt{6-\sqrt{11}} - \sqrt{22}$. Then

   (a) $n \geq 1$ **(b)** $0 < n < 1$ **(c)** $n = 0$ **(d)** $-1 < n < 0$

   (e) $n \leq -1$

4. In the figure two congruent 30°-60° right triangles, AOB and CPD are drawn so that AO is opposite to the 60° angle in AOB and so that the points O and P trisect the segment $\overline{AC}$. If the length of $\overline{DP}$ is one unit, find the perimeter of the quadrilateral $ABCD$ (not drawn).



   (a) $2(2+\sqrt{3})$ **(b)** $2(3+\sqrt{2})$ **(c)** $4+2\sqrt{2}$ **(d)** $4+\sqrt{7}$

   (e) None of the above

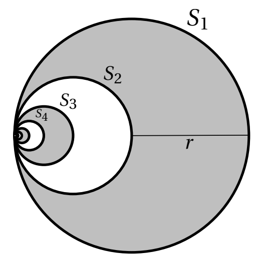
5. Find the number of 6-digit numbers that use only the digits 1 and 9, but do not have two 1's next to each other.

   (a) 19 **(b)** 21 **(c)** 49 **(d)** 14 **(e)** None of the above

6. The sum of the solutions of the equation $\log_2\left(x^2 - 8x - 8\right) = 0$ is

   (a) 9 **(b)** 10 **(c)** -8 **(d)** 8 **(e)** None of the above

7. Let $S_1, S_2, S_3, \ldots$ be a sequence of nested circles such that S_{i+1} is inside S_i and the radius of S_{i+1} is half of that of S_i. Let r denote the radius of S_1. For $i = 1, 3, 5, \ldots$ (i.e. for odd i's), the region inside S_i and outside S_{i+1} is shaded. Find the sum of the shaded areas.



   (a) $\dfrac{3\pi r^2}{4}$

   (b) $\dfrac{\pi r^2}{4}$ **(c)** $\dfrac{4\pi r^2}{5}$ **(d)** $3\pi r^2$ **(e)** None of the above

8. For which values of a is $(\log_a 7)(\log_7 6) = \ln 36$?

   (a) $\frac{1}{e}$ **(b)** 6 **(c)** $\sqrt{e}$ **(d)** 2 **(e)** None of the above

9. For which x is $2\sin^{-1}\left(\sin\dfrac{x}{2}\right) = x$?

   (a) For all real x **(b)** For $-1 \le x \le 1$ **(c)** For $-\pi \le x \le \pi$

   (d) For $-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$ **(e)** None of the above

10. At exactly 3:22, what is the angle between the hour hand and the minute hand of a clock? (Assume that both hands rotate in smooth continuous motion without jumps.)

    (a) 35° **(b)** 7° **(c)** 31° **(d)** 42° **(e)** None of the above

11. For which real values of p does the graph of $y = x^2 - 2px + p + 1$ have exactly one x-intercept?

    (a) No such p exists **(b)** $p = 0$ **(c)** $p = \dfrac{1+\sqrt{5}}{2}$ only

    (d) $p = \frac{1}{2}$ **(e)** None of the above

12. A committee of 4 people is to be randomly chosen from a group of 7 people, including Mr. and Mrs. Smith. What is the probability that a committee will not contain both Mr. Smith and Mrs. Smith?

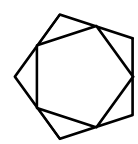
    (a) $\frac{5}{7}$ **(b)** $\frac{1}{3}$ **(c)** $\frac{4}{7}$ **(d)** $\frac{2}{3}$ **(e)** None of the above

13. $\cos\left(2\cos^{-1}\left(-\frac{12}{13}\right)\right) =$

    (a) Undefined **(b)** $\frac{119}{169}$ **(c)** $-\frac{24}{13}$ **(d)** $-\frac{119}{169}$

    (e) None of the above

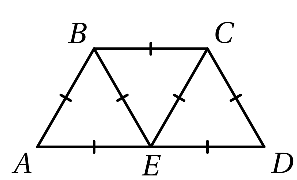
14. Let θ be the angle measure of a regular polygon with n vertices, where n is a fixed natural number larger than 2. One such polygon is inscribed in another so that the vertices of the inner polygon are the midpoints of the edges of the outer one (the figure gives an example for $n = 5$). If the side of the outer polygon is a units and that of the inner polygon is b units, which of the following is true?



    (a) $\cos\theta = \dfrac{a^2 - 2b^2}{a^2}$ **(b)** $\cos\theta = \dfrac{2a^2 - b^2}{2a^2}$ **(c)** $\sin(\theta/2) = \dfrac{a}{2b}$

    (d) $\tan\theta = \dfrac{b}{\sqrt{a^2 - b^2}}$ **(e)** None of the above

15. Given that the trapezoid $ABCD$ has area 12, find the length of the segment $\overline{AE}$. (Congruent line segments are marked with small perpendicular dashes through the middle of the segment).



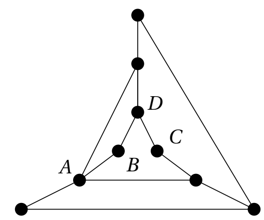
    (a) $\sqrt{6}$ **(b)** $\frac{4}{\sqrt{3}}$ **(c)** $\frac{4}{\sqrt[4]{3}}$ **(d)** 6

    (e) None of the above

16. Suppose $f(1) = 2$ and $f(5) = 3$. If for every $k \geq 3$, $f(k) = f(k-2) - f(k-1)$, find $f(2)$.

    (a) $\frac{7}{5}$ **(b)** $\frac{5}{2}$ **(c)** -3 **(d)** $\frac{1}{3}$ **(e)** None of the above

17. A peg located at one vertex on the corresponding graph will be moved to another vertex that is connected to its current location by an edge; the selection of the new position is otherwise totally random. If the peg is currently at the vertex A, what is the probability that it will end up after two moves at an interior vertex (one of vertices B, C or D)?



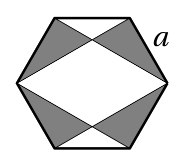
    (a) 7/24 **(b)** 1/2 **(c)** 3/10 **(d)** 5/24

    (e) None of the above

18. Suppose that $f(x) \geq 0$ for all x, and a, b are positive numbers. Given that the area of the region bounded by the graph of $y = f(x)\sin(x)$, $0 \leq x \leq \pi$, and the x-axis is 2 units, and that the area bounded by the graph $y = af(bx)\sin(bx)$, $0 \leq x \leq \pi/b$, and the x-axis is 8 units, which of the following holds?

    (a) $a = 4$ **(b)** $a = 2$ **(c)** $\frac{a}{b} = 4$ **(d)** $\frac{b}{a} = 4$ **(e)** None of the above

19. In the figure is a regular hexagon of side length a. The ratio of the shaded area to that of the entire hexagon is equal to



    (a) $\frac{1}{3}$ **(b)** $\frac{2}{3}$ **(c)** $\frac{4}{9}$ **(d)** $\frac{1}{4}$ **(e)** None of the above

20. Let a be a positive real number and k a natural number. Let f be a polynomial of degree 11 with a positive leading coefficient whose roots are $a, a\pm1, a\pm2, a\pm3, a\pm4, a\pm5$, and let $g(T) = (f(T))^2$. If the leading coefficient of g is a while the coefficient of T^{21} in g is $-44a$, what is $f(0)$?

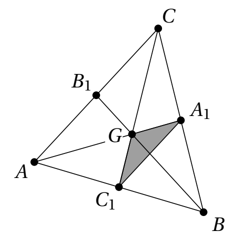
    (a) 0 **(b)** 11 **(c)** 1 **(d)** 11! **(e)** None of the above

21. Let $a = 5^{39}$, $b = 2^{62}$, $c = 6^{26}$, and $d = 3^{52}$. Which of the following inequalities holds?

(a) $b < c < a < d$ **(b)** $c < a < d < b$ **(c)** $c < d < b < a$

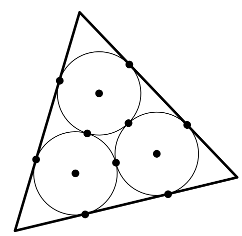
(d) $b < c < d < a$ **(e)** None of the above

22. Assume that the area of triangle ABC is $4\,\text{ft}^2$. Let A_1, B_1 and C_1 be the midpoints of the sides BC, AC and AB, respectively, and let G be the point of intersection of AA_1, BB_1 and CC_1. Then the area of triangle A_1C_1G is



(a) $\frac{4}{9}$ **(b)** $\frac{1}{3}$ **(c)** $\frac{1}{12}$ **(d)** $\frac{1}{16}$ **(e)** None of the above

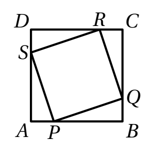
23. Three circles with radius 3 are inscribed in an equilateral triangle as shown in the picture. The side of the triangle is



(a) 12 **(b)** $6\left(1+\sqrt{3}\right)$

(c) $3+3\sqrt{3}$ **(d)** 18 **(e)** None of the above

24. Let $ABCD$ be a square with a side a and let the points P, Q, R, S be on the sides AB, BC, CD, DA respectively, and such that $AP = BQ = CR = DS = \frac{1}{3}a$. Let $PQ = b$. Then $\dfrac{a}{b}$ is



(a) $\frac{3}{5}$ **(b)** $\frac{1}{3\sqrt{5}}$ **(c)** $\frac{5}{\sqrt{3}}$ **(d)** $\frac{3}{\sqrt{5}}$ **(e)** None of the above

25. The expression $4\sin^3\alpha - 3\sin\alpha + \sin 3\alpha$ equals

(a) -2 **(b)** 0 **(c)** 2 **(d)** $6\sin\alpha$ **(e)** None of the above