Saginaw Valley State University

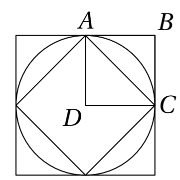
2006 Math Olympics – Level I Solutions

1. A square is inscribed in a circle, which is inscribed in another square. What is the ratio of the area of the smaller square to the area of the larger square?

(a) $\frac{1}{2}$ **(b)** $\frac{1}{\sqrt{2}}$ **(c)** $\frac{1}{4}$ **(d)** Impossible to determine from the information given

(e) None of the above

SOLUTION **(a)**: Position the squares as shown in the picture. Then the area of the large square is 4 times the area of the square $ABCD$, while the area of the small square is 4 times the area of the triangle ACD. Since the area of the triangle is exactly 1/2 of the area of the square $ABCD$, the ratio of the area of the smaller square to the area of the larger square is 1/2.



2. Which of the following choices has a different value from the other choices?

(a) $32 \cdot 2^{35}$ **(b)** $(32)^8$ **(c)** $\dfrac{2^{40}+2}{2}$ **(d)** $2^{39}+2^{39}$ **(e)** $\dfrac{2^{45}+2^{45}}{2^5+2^5}$

SOLUTION **(c)**: The choice c is the only choice which is odd. (All the other choices are equal to 2^{40}).

3. At what time between 7:00pm and 8:00pm will the hour hand and the minute hand point in opposite directions? (Assume that the hands move smoothly without jumps.)

(a) 7 minutes after 7:00pm.

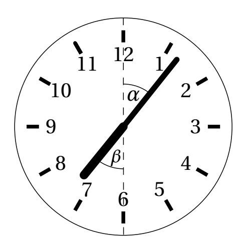
(b) $6\,{}^{5}/_{12}$ minutes after 7:00pm.

(c) $5\,{}^{5}/_{11}$ minutes after 7:00pm.

(d) There is no time between 7:00pm and 8:00pm at which the hour hand and the minute hand point in opposite directions.

(e) None of the above

SOLUTION **(c)**: Since the hour hand will be between 7 and 8, the minute hand must be between 1 and 2. The angle α between the minute hand and 12 has to be the same as the angle between the hour hand and 6. Each minute, the minute hand angle increases by 6°, while the hour hand inreases by $30/60 = 1/2$ degree. At exactly 7:00, $\alpha = 0$ while $\beta = 30°$. Then, t minutes after 7:00, $\alpha = 6t$ and $\beta = 30 + .5t$ degrees. When both angles are equal, we get $6t = 30 + .5t$, or $11t = 60$, or $t = 5\,{}^{5}/_{11}$.



4. The expression $\dfrac{x^2+2xy-2yz-z^2}{x+2y+z}$ simplifies to

(a) $2x-2y-z^2$ **(b)** $x-z$ **(c)** $2x-2y-z$

(d) $x-2y-z$ **(e)** None of the above

SOLUTION **(b)**:

$$\begin{aligned}\frac{x^2+2xy-2yz-z^2}{x+2y+z} &= \frac{x^2-z^2+2y(x-z)}{x+z+2y}\\ &= \frac{(x+z)(x-z)+2y(x-z)}{x+z+2y}\\ &= \frac{(x+z+2y)(x-z)}{x+z+2y}\\ &= x-z\end{aligned}$$

5. Which of the following definitions of $x ✿ y$ guarantees that $x ✿ y = y ✿ x$?

(a) $x ✿ y = \dfrac{x}{y}$ **(b)** $x ✿ y = 2x+y$ **(c)** $x ✿ y = \dfrac{x}{y}+\dfrac{y}{x}$

(d) $x ✿ y = x^2y$ **(e)** None of the above

SOLUTION **(c)**:

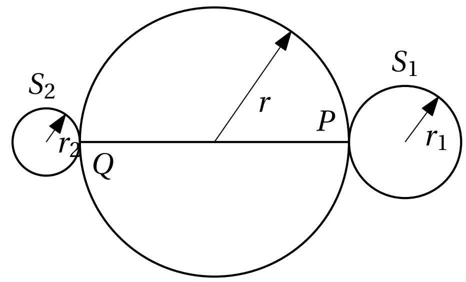
$$x ✿ y = \frac{x}{y}+\frac{y}{x} = \frac{y}{x}+\frac{x}{y} = y ✿ x$$

6. A kindergarten teacher needs to seat 5 children in a row. The twins, Terry and Sherry, need to sit next to each other or they will cry. How many ways can the teacher seat the 5 children so that Terry and Sherry are next to each other?

(a) 125 **(b)** 48 **(c)** 24 **(d)** 12 **(e)** None of the above

SOLUTION **(b)**: We can start by considering the twins as one person, since they have to be next to each other. That means we have 4 people to seat, which gives us 4! = 24 different ways. But then for each of those 24 ways, we can have Terry on the left and Sherry on the right, or Sherry on the left and Terry on the right, so altogether we have 48 ways.

7. The two smaller circles S_1 and S_2 of radii r_1 and r_2 are initially positioned, respectively, at points P and Q at the opposite ends of a diameter of the big circle of radius r. The smaller circles are then moved along the perimeter of the big circle by rotating them clock-wise (without skidding) at the same rotational rate. After completing 5.5 full rotations, S_1 reaches Q, while at that moment S_2 is only two-thirds of the distance along the arc from Q to P. If the circles continue to rotate until the first moment they are back (simultaneously) to their initial positions, how many rotations would S_2 have made?



(a) 27 **(b)** 16.5 **(c)** 70.5 **(d)** 141 **(e)** None of the above

SOLUTION **(e)**: After another 5.5 rotations, S_1 will be back at its initial position, while S_2 will be one third of the way from P to Q. Next 5.5 rotations will place S_2 back to its initial position, while S_1 will be at at Q (in other words, both circles will be at the same point). We need to repeat this whole process again for the circles to return to their original positions. That means that the number of rotations will be $6 \cdot 5.5 = 33$ rotations.

8. $a \circledast b$ is defined to be $a + \dfrac{1}{b}$ for $b \neq 0$. Find $(1 \circledast 2) \circledast \left(2 \circledast \frac{1}{4}\right)$.

(a) $\frac{5}{3}$ **(b)** $\frac{15}{4}$ **(c)** 2 **(d)** $\frac{10}{9}$ **(e)** None of the above

SOLUTION **(a)**:

$$\begin{aligned}(1 \circledast 2) \circledast \left(2 \circledast \frac{1}{4}\right) &= \left(1 + \frac{1}{2}\right) \circledast \left(2 + \frac{1}{1/4}\right) \\ &= \left(1 + \frac{1}{2}\right) + \frac{1}{2 + \frac{1}{1/4}} \\ &= 1 + \frac{1}{2} + \frac{1}{2+4} \\ &= \frac{10}{6} = \frac{5}{3}\end{aligned}$$

9. What is the sum of the solutions to the equation $\dfrac{x}{x-1} = 4x^2 + \dfrac{1}{x-1}$?

(a) 1 **(b)** 1.5 **(c)** 0 **(d)** $\frac{1}{2}$ **(e)** None of the above

SOLUTION **(c)**: Multiply the equation by $x - 1$ to get $x = 4x^2(x-1) + 1$ or $x = 4x^3 - 4x^2 + 1$ or $(4x^2 - 1)(x+1) = 0$ which has solutions 1, $\pm 1/2$. However, 1 cannot be a solution of the original equation (division by 0), so the sum of the solutions is $\frac{1}{2} - \frac{1}{2} = 0$.

10. A lottery game uses a bowl filled with red and white ping-pong balls. Some of the balls have a black dot and some do not. 50% of the balls are colored red, and 20% of the red balls have a black dot. 40% of all the balls have a black dot. What percent of the balls will be white with a black dot?

    (a) 20% **(b)** 60% **(c)** 30% **(d)** 80% **(e)** None of the above

    SOLUTION **(c)**: Assume there are exactly 100 balls. Then 50 of them are red, and 40 of all the balls have a black dot. Out of the 50 red balls, 20%, which is $.2 * 50 = 10$, have black dot. So out of the 40 balls with black dot, 10 are red, which leaves 30 white. So out of the 100 balls, 30 are white with black dot, which is 30%.

11. The equation of the tangent line to the circle $x^2 + y^2 = 25$ at $(3,4)$ is

    (a) $3x+4y=25$ **(b)** $3x-4y=-7$ **(c)** $4x+3y=24$

    (d) $4x-3y=0$ **(e)** None of the above

    SOLUTION **(a)**: The circle has center at $(0,0)$, so the tangent line to the circle at the point $(3,4)$ must be perpendicular to the line through $(0,0)$ and $(3,4)$. The slope of the line through $(0,0)$ and $(3,4)$ is

    $$\frac{4-0}{3-0} = \frac{4}{3}$$

    so the slope of the tangent line must be $-\frac{3}{4}$. The tangent line has to pass through the point $(3,4)$, so the point-slope equation of the tangent line will be $y-4=-\frac{3}{4}(x-3)$. Multiplying by 4 and adding $3x+16$ to both sides gives us $3x+4y=25$.

12. A worker is pleased to receive a 20% pay rise, but disappointed to have this followed by a 20% pay cut. What is the overall result of this?

    (a) The pay remains the same. **(b)** 10% cut

    (c) 4% rise **(d)** 4% cut

    (e) None of the above

    SOLUTION **(d)**: If the original salary was x, the salary after the rise is $1.20x$, and the salary after the cut is $.8 \cdot 1.20x = .96x$, which is equivalent to 4% cut.

13. If a cat and a half can catch a rat and a half in an hour and a half, how many rats can 3 cats catch in 3 hours?

    (a) 9 **(b)** 6 **(c)** 3 **(d)** 2 **(e)** None of the above

    SOLUTION **(b)**: First we double the number of cats, which will double the number of rats caught. Then we double the number of hours, which will double the number of rats caught again. We get $1.5 \cdot 2 \cdot 2 = 6$.

14. Donald rents ponies for \$15 for one pony. For each additional pony, a \$1 per pony discount is given (e.g. 2 ponies rent for \$14 each, 3 ponies rent for \$13 each etc). What is the most expensive rental Donald can have?

(a) \$225 **(b)** \$64 **(c)** \$49 **(d)** \$144 **(e)** None of the above

SOLUTION **(b)**: Donald's revenue for n ponies is $n(16-n) = 16n - n^2$. That is a quadratic function with maximum at $n = 8$, and the revenue for b ponies is $8 \cdot 8 = 64$.

15. For $0 < x < 2$, the minimum value of $x + \dfrac{1}{x}$ is

(a) $\dfrac{3}{4}$ **(b)** 1 **(c)** 2 **(d)** The expression has no minimum value

(e) None of the above

SOLUTION **(c)**: Rewrite the expression like so:

$$x + \frac{1}{x} = \frac{x^2+1}{x} = \frac{x^2-2x+1+2x}{x} = \frac{(x-1)^2}{x} + 2$$

Since $x > 0$, $\frac{(x-1)^2}{x}$ is always positive or 0, so it's minimal value is 0 for $x = 1$. Therefore the minimal value of $x + 1/x$ is $0 + 2$.

16. How many gallons of pure alcohol must be added to 10 gallons of a 40% alcohol mixture to get a 50% alcohol mixture?

(a) 10 **(b)** 5 **(c)** 3 **(d)** 2 **(e)** None of the above

SOLUTION **(d)**:

Mixture	**Concentration**	**Amount**	**Amount of alcohol**
Pure alcohol	100%	x gal	x
40% mixture	40%	10 gal	$0.4 \cdot 10$ gal
50% mixture	50%	$10 + x$ gal	$0.5(10 + x)$ gal

From the last column of this table we get the equation $x+4 = 0.5(10+x)$, or $x+4 = 5+0.5x$, or $x = 2$.

17. A boat can go upstream against a 2.5 mph current in 11 hours, and back downstream to the same place in 7 hours. Find the speed of the boat in still water.

(a) 5 mph **(b)** 7 mph **(c)** 9 mph **(d)** 13 mph **(e)** None of the above

SOLUTION **(e)**: Let x be the speed of the boat in still water. Since distance = rate × time, and the distances are equal, $11(x-2.5) = 7(x+2.5)$, or $11x - 27.5 = 7x + 17.5$, or $4x = 45$, or $x = 11.25$.

18. A tourist travels to Canada and exchanges 1000 American dollars, getting 1250 Canadian dollars. After 4 hours in a casino, he has 100 Canadian dollars left. Assuming the exchange rate did not change during his stay, how much will he get exchanging it for American dollars?

    (a) 75 **(b)** 80 **(c)** 85 **(d)** 90 **(e)** None of the above

    SOLUTION **(b)**:

$$\frac{1000 \text{ US dollars}}{1250 \text{ Canadian dollars}} = 1$$

which means that

$$100 \text{ Canadian dollars} = 100 \text{ Canadian dollars } \times \frac{1000 \text{ US dollars}}{1250 \text{ Canadian dollars}}$$

which is 80 US dollars.

19. On Halloween, Tom collects twice as many pieces of candy as Jane, while Mary gets three times as many pieces as Rich. Tom eats half of his candy and gives the other half to Mary. Mary consumes two-thirds what she now has and asks Jane if she wants the rest. Jane opts to take from Mary only as much as Jane originally had. Mary then passes the remaining candy to Rich. Rich ends up with 100 pieces of candy, which is, according to him, 25% more than he originally collected. Which of the following holds?

    (a) The kids collected 550 pieces of candy altogether

    (b) Mary and Jane together collected more than 130% of what Tom and Rich collected together

    (c) Tom and Jane together originally collected 270 pieces

    (d) Mary ended up consuming more candy than she collected

    (e) None of the above

    SOLUTION **(c)**: Letting T, J, M, and R stand for the number of pieces of candy collected by Tom, Jane, Mary, and Rich respectively, we have $T = 2J$ and $M = 3R$. But we also know $100 = 1.25R$ according to the last sentence, so $R = 80$, which also implies $M = 240$. The amount that Mary gives to Rich is $\frac{1}{3}(M + \frac{1}{2}T) - J$. So $\frac{1}{3}(M + \frac{1}{2}T) - J + R = 100$. Putting in $R = 80$, $M = 240$, and $T = 2J$ and simplifying gives $160 - \frac{2}{3}J = 100$. So $J = 90$, $T = 180$, and $J + T = 270$.

20. What is the equation of the perpendicular bisector of the line segment $(-1,3)$ to $(3,5)$? (A line bisects a segment if it divides it into two segments of equal length.)

(a) $x-2y=-7$ **(b)** $2x-y=-2$ **(c)** $x+2y=13$

(d) $2x+y=1$ **(e)** None of the above

SOLUTION **(e)**: The slope of the line through $(-1,3)$ and $(3,5)$ is

$$\frac{5-3}{3-(-1)}=\frac{2}{4}=\frac{1}{2}.$$

The slope of the perpendicular bisector is then -2. The bisector has to pass through the midpoint of the two given points:

$$\left(\frac{-1+3}{2},\frac{3+5}{2}\right)=(1,4)$$

Out of the given lines, $2x+y=1$ is the only one with slope -2, and since $2\cdot 1+4=6\neq 1$, this line does not pass through the midpoint. So none of the given lines is right.

21. What is the remainder when $9\cdot 10^5+8\cdot 10^4+7\cdot 10^3+6\cdot 10^2+5\cdot 10^1+4\cdot 10^0$ is divided by 3?

(a) 0 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** None of the above

SOLUTION **(a)**: The number is actually 987654 which is divisible by 3, since the sum of the digits is divisible by this number.

22. Let a be a fixed integer. Alowing repetition, let r_1, r_2 and r_3 be the solutions of the equation $x^3+ax^2+ax+1=0$. If r_1 and r_2 are distinct integers, what is the value of r_3?

(a) $\sqrt[3]{3}$ **(b)** 1 **(c)** $-\frac{1}{2}$ **(d)** $1+i$ **(e)** None of the above

SOLUTION **(b)**: The only rational solution of the equation can be ± 1, therefore $\{r_1,r_2\}=\{\pm 1\}$ and $a=-1$. Then the remaining solution, r_3, is 1.

23. In a strange far-away land, the "star operation" misbehaves: $a\star b\neq b\star a$. Other than that, the operation is associative and $1\star a=a\star 1=a$ for any a. Suppose we know that $a\star a\star a=1$, $b\star b=1$, and $b\star a=a\star a\star b$. Which of the following must be true about $b\star a\star a$?

(a) $b\star a\star a=1$ **(b)** $b\star a\star a=a\star b$ **(c)** $b\star a\star a=a\star a\star b$

(d) $b\star a\star a=a$ **(e)** None of the above

SOLUTION **(b)**: $b\star a\star a=(b\star a)\star a=(a\star a\star b)\star a=(a\star a)\star(b\star a)=(a\star a)\star(a\star a\star b)=(a\star a\star a)\star(a\star b)=1\star(a\star b)=(a\star b)$

24. The solution set to the equation $2x^3 - 9x^2 + 25 = 0$ is

(a) $\{-\frac{5}{2}, 1+\sqrt{6}, 7-2\sqrt{6}\}$ **(b)** $\{\frac{5}{2}, 1-\sqrt{6}, 7-2\sqrt{6}\}$ **(c)** $\{\frac{5}{2}, 1-\sqrt{6}, 7+2\sqrt{6}\}$

(d) $\{-\frac{5}{2}, 1-\sqrt{6}, 7+\sqrt{6}\}$ **(e)** None of the above

SOLUTION **(e)**: Using the Rational Zero theorem and synthetic division, we can easily discover that the equation has a rational solution $5/2$, with the factored form $(2x-5)(x^2-2x-5) = 0$. By completing the square, the quadratic factor becomes $(x-1)^2 - 6$, so the solution set is

$$\left\{-\frac{5}{2}, 1-\sqrt{6}, 1+\sqrt{6}\right\}.$$

You could also notice that as the quadratic factor has rational coefficients, the two irrational solutions from the quadratic factor have to be conjugate. None of the proposed solution sets has conjugate irrational solutions.

25. Let $n = \sqrt{6+\sqrt{11}} + \sqrt{6-\sqrt{11}} - \sqrt{22}$. Then

(a) $n \geq 1$ **(b)** $0 < n < 1$ **(c)** $n = 0$ **(d)** $-1 < n < 0$ **(e)** $n \leq -1$

SOLUTION **(c)**: We need to compare the numbers $a = \sqrt{6+\sqrt{11}} + \sqrt{6-\sqrt{11}}$ and $b = \sqrt{22}$. Since both numbers are obviously positive, we can compare their squares instead.

$$\begin{aligned} a^2 &= \left(\sqrt{6+\sqrt{11}} + \sqrt{6-\sqrt{11}}\right)^2 \\ &= 6+\sqrt{11} + 2\sqrt{6+\sqrt{11}}\sqrt{6-\sqrt{11}} + 6 - \sqrt{11} \\ &= 12 + 2\sqrt{\left(6+\sqrt{11}\right)\left(6-\sqrt{11}\right)} \\ &= 12 + 2\sqrt{36-11} \\ &= 22 \end{aligned}$$

Therefore $a^2 = 22 = b^2$, therefore $a = b$ and $n = 0$.