Saginaw Valley State University

2005 Math Olympics – Level II

1. If the point $(3, 5)$ is on the graph of $y = -3f(2x + 1) + 2$, which of the following must be true?

   (a) $f(7) = 13$ **(b)** $f(2) = 13$ **(c)** $f(2) = -1$ **(d)** $f(7) = -1$

   (e) None of the above

   SOLUTION. **D**

   Pluging in for x and y, we get $5 = -3f(2 \cdot 3 + 1) + 2$, or $5 = -3f(7) + 2$, or $3 = -3f(7)$ or $-1 = f(7)$.

2. Let p be a prime number and k an integer such that $x^2 + kx + p = 0$ has two distinct positive integer solutions. The value of $k + p$ is

   (a) 1 **(b)** -1 **(c)** 0 **(d)** 2 **(e)** -2

   SOLUTION. **B**

   Let a and b be the two distinct positive integer solutions. Then $x^2 + kx + p = (x - a)(x - b)$ and $k = -a - b$ while $p = ab$. Because p is a prime, the only possible values for a and b are 1 and p. We can assume that $a = 1$ and $b = p$. So $k + p = -a - b + p = -1 - p + p = -1$.

3. Which of the following are equivalent to $\log(\csc^2 x)$?

   (a) $\dfrac{1}{\log^2(\sin x)}$ **(b)** $\dfrac{1}{\log^2(\cos x)}$ **(c)** $-2\log(\sin x)$

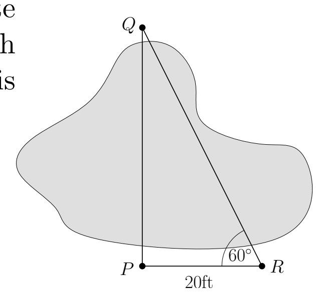
   (d) $\log(2 \csc x)$ **(e)** None of the above

   SOLUTION. **C**

   Using rules of logarithms and the definition of $\csc x$:

$$\log\left(\csc^2 x\right) = \log\left(\frac{1}{\sin^2 x}\right) = \log 1 - \log\left(\sin^2 x\right) = 0 - 2\log\left(\sin x\right)$$

4. To find the distance between two points P and Q on opposite sides of a lake, a surveyor locates a point R, 20 feet from P, such the line PR is perpendicular to the line PQ. The angle PRQ is $60°$. Find the distance between P and Q.



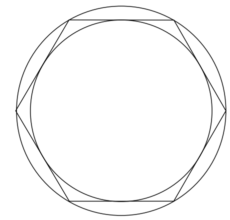
   (a) $20\sqrt{3}$ ft **(b)** 40 ft **(c)** $\dfrac{20\sqrt{3}}{3}$ ft

   (d) 30 ft **(e)** None of the above

   SOLUTION. **A**

   $\triangle PQR$ is a (30,60)-right triangle. Hence, $PQ = PR \tan 60° = 20\sqrt{3}$.

5. The difference between the area of the circle inscribed (the inner circle) in a regular hexagon of side length $\sqrt{3}$ and the area of the circle in which the hexagon is inscribed (the outer circle) is

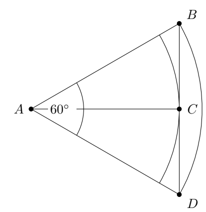


(a) $\pi/2$ **(b)** $3\pi/2$ **(c)** $3\pi/4$ **(d)** $2\pi/3$

(e) None of the above

SOLUTION. **C**

The angle BAD is $360°/6 = 60°$, and the triangle is isosceles. That means that the two other angles in the triangle are also $60°$, the triangle is actually equilateral, and so all the sides of the triangle have the same lenths. From this it follows that the radius of the larger circle is $\sqrt{3}$. Using the right triangle ABC, we can easilly see (Pythagorean Treorem or trigonometry) that the length of AC, which is the radius of the smaller circle, is $\frac{3}{2}$. So the difference of the areas of the two circles is $\pi\left(\left(\sqrt{3}\right)^2 - \left(\frac{3}{2}\right)^2\right) = \pi\left(3 - \frac{9}{4}\right) = \frac{3\pi}{4}$.



6. The product N of three positive integers is six times their sum, and one of the integers is the sum of the other two. Find the sum of all possible values of N

(a) 300 **(b)** 350 **(c)** 336 **(d)** 318 **(e)** None of the above

SOLUTION. **C**

According to the proplem, we have positive integers a, b such that $a \cdot b \cdot (a+b) = N = 6(a+b+(a+b)) = 12(a+b)$. So, $a \cdot b = 12$. This gives the following possibilities:

a	b	$a+b$	N
12	1	13	156
6	2	8	96
4	3	7	84

Thus, the sum of all possible values of N is $156 + 96 + 84 = 336$.

7. Suppose that the digits 1, 0, and 8 are written so that they look the same when they are upside down as they do when they are right-side up. Also, the digits 6 and 9 are written so that by turning (rotating) a 6 upside down, we get a 9. Two 5-digit numbers are "flips" of each other if one is obtained by rotating the other upside down. For example, 61891 and 16819 are "flips" of each other. If all five-digit numbers (including those starting with 0 like 00027) are being printed on slips of paper such that two numbers are printed on the same slip if and only if they are flips of one another, how many slips of paper are needed?

(a) 100,000 **(b)** 98475 **(c)** 96875 **(d)** $10^5 - \dfrac{5^5}{2}$

(e) None of the above

SOLUTION. **B**

The correct count is: #(5-digit numbers) - $\frac{1}{2}$ #(numbers whose "flip" is a *different* but *meaningful* number) There are $10^5 = 100000$ five-digit numbers. Numbers that flip to a *meaningful* (but not necessarily different) number have 0, 1, 6, 8, or 9 for its five digits, hence there are 5^5 such numbers. Of these, some are the "flips"' of themselves. These will have has 0, 1, 6, 8, or 9 for for the fist two places (which consequently determine the last two digits), and 0, 1 or 8 for the middle (third) digit. Hence there are $5 \times 5 \times 3$ numbers. So, there are

$5^5 - 3 \cdot 5^2 = 3050$ numbers whose "flip" is a *different* but *meaningful* number. So the number of slips of paper will be $10^5 - \frac{1}{2}(3050) = 100,000 - 1525 = 98475$.

8. Which of the following is not equivalent to $\sin(15°)$?

(a) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ **(b)** $\dfrac{\sqrt{2-\sqrt{3}}}{2}$ **(c)** $\dfrac{1}{\sqrt{6}+\sqrt{2}}$

(d) $\dfrac{2-\sqrt{3}}{4}$ **(e)** $\tan^{-1}\left(\dfrac{2-\sqrt{3}}{2+\sqrt{3}}\right)$

SOLUTION. **D**

Upon close examination, you can see that the option in (d) is the square of the one in (b), and since they are not equal to 1, they cannot be both equivalent to the same thing. So one of them must be the answer. Using the half angle formula, we get

$$\sin(15°) = \sqrt{\frac{1-\cos(30°)}{2}} = \sqrt{\frac{\frac{2}{2}-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

Added Remark: Choice (b) equals $\sin 15°$ *by the half-angle formula. For (e),* $\sin 15° = \sin(75-60)$*; now use the difference formula of sine. (a) and (c) are clearly equivalent, just rationalize the numerator of (a) and simplify. Finally, (a) and (b) are seen equivalent as follows* $\sqrt{2-\sqrt{3}} = \frac{1}{2}\sqrt{8-4\sqrt{3}} = \frac{1}{2}\sqrt{(6+2)-2\sqrt{12}} = \frac{1}{2}\sqrt{(\sqrt{6}-\sqrt{2})^2} = \frac{1}{2}(\sqrt{6}-\sqrt{2})$.

9. Let $x☼y = \frac{x}{y} + \frac{y}{x}$. If a is a positive number such that $a☼(a+1) = \frac{5}{2}$, which of the below gives the value of a?

(a) 1 **(b)** 2

(c) $\sqrt{5}$ **(d)** there is no such number

(e) None of the above

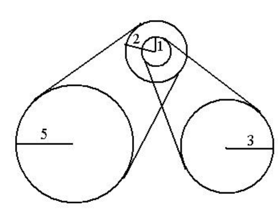
SOLUTION. **A**

By the definition of ☼, we get $a☼(a+1) = \frac{a}{a+1} + \frac{a+1}{a} = \frac{a^2+(a+1)^2}{a(a+1)} = \frac{2a^2+2a+1}{a^2+a}$. So,

$$\frac{5}{2} = \frac{2a^2+2a+1}{a^2+a}$$

By cross-multiplying we get $5a^2 + 5a = 4a^2 + 4a + 2$ or $a^2 + a - 2 = 0$. This equation has solutions 1 and -2. The only *positive* solution is 1.

10. A belt driven wheel system shown in the figure consists of two outside simple wheels of radii 5in and 3in, respectively, and a middle compound wheel made by gluing concentric 2in and 1in wheels, so that the two wheels will rotate as one unit. Two belts are attached tightly between the wheels as shown. If the 5in wheel is moving at 4 rotations per minute (rpm), how fast is the 3in wheel rotating?



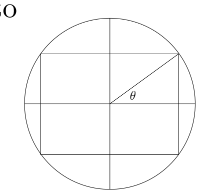
(a) 20/3 rpm **(b)** 20 rpm **(c)** 24 rpm **(d)** 30 rpm

(e) None of the above

SOLUTION. **E**

The ratio of the rotational speeds of two wheels connected by a belt is equal to ratio of their perimeters (= ratio of their radii). The two discs of the (middle) compound wheels rotate at the same speed. Hence speed of the compound wheel is $4 \times \frac{5}{2} = 10$ rpm, and the speed of the 3in wheel is $10 \times \frac{1}{3} = \frac{10}{3}$.

11. A rectangle in inscribed in a circle of radius 1. The angle from the center to the corner A is θ. The area of the rectangle equals



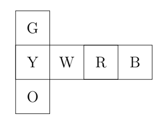
(a) $\cos^2\theta - \sin^2\theta$ **(b)** $\sin 2\theta$ **(c)** $2\sin 2\theta$

(d) $2\sin\theta\cos\theta$ **(e)** None of the above

SOLUTION. **C**

The sides of the rectangle are $2\sin\theta$ and $2\cos\theta$. So the area is $4\sin\theta\cos\theta = 2\sin 2\theta$.

12. A butterfly lands on one of the six squares of the t-shaped figure shown and then randomly moves to an adjacent square. What is the probability that the butterfly ends up on the R square.



(a) 1/2 **(b)** 1/3 **(c)** 2/3 **(d)** 1/6 **(e)** None of the above

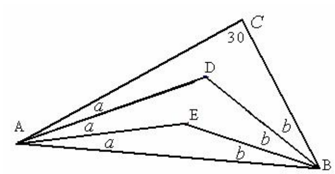
SOLUTION. **C**

Pr(Ending on R in the second step)

= Pr(landing on B then move to R) + Pr(landing on W first then move to R)

$= \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{4}$

13. The corresponding trisectors of two angles (A and B) of a scalene triangle meet at points D and E. The third angle of the triangle (angle C) is 30 degrees. Find the measure of angle D.



(a) 130 **(b)** 80 **(c)** 120 **(d)** 90

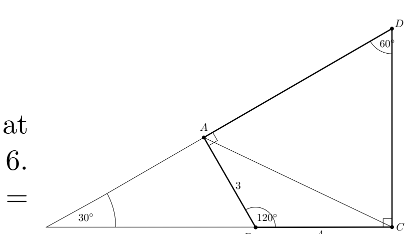
(e) None of the above

SOLUTION. **B**

Since the sum of angles in any triangle is 180°, we know that $mCAB + mABC = 150°$, or, written using a and b, $3a + 3b = 150°$. Therefore $mDAB + mABD = 2a + 2b = \frac{2}{3}150° = 100°$, and so the measure of the angle D must be 80°.

14. In ΔABC, $m\angle ABC = 120°$, $AB = 3$ and $BC = 4$. If perpendiculars constructed to $\overline{AB}$ at A and to $\overline{BC}$ at C meet at D, then $CD =$

(a) 5 **(b)** $\dfrac{8}{\sqrt{3}}$ **(c)** $\dfrac{11}{2}$ **(d)** $\dfrac{10}{\sqrt{3}}$ **(e)** None of the above



SOLUTION. **D**

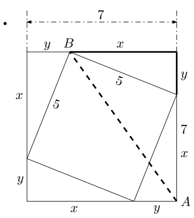
In construction in the problem, we extended $\overline{AD}$ and $\overline{BC}$ so they meet at F. Then $m\angle DFC = 30°$. In the right $\triangle FAB$, $\overline{FB} = \overline{AB}/\sin(30°) = 6$. Hence $\overline{FC} = \overline{FB} + \overline{BC} = 6 + 4 = 10$. Now, in the right $\triangle FCD$, $\overline{CD} = \overline{FC}\tan(30°) = 10/\sqrt{3}$.

15. A square of perimeter 20 is inscribed in a square of perimeter 28 (inscribed means the vertices of the smaller square are on the sides of the larger square). What is the greatest distance between a vertex of the inner square and a vertex of the outer square?

(a) $5\sqrt{3}$ **(b)** $\sqrt{58}$ **(c)** $\frac{7\sqrt{5}}{2}$ **(d)** $\sqrt{65}$ **(e)** None of the above

Solution. **D**

The side length of the outer square is 7 and that of the inner one is 5. The corners of the inner square divide each side of the outer square into two segments of lengths x and y as in the figure. Without loss of generality assume $x \geq y$. So we have the following system of equations:

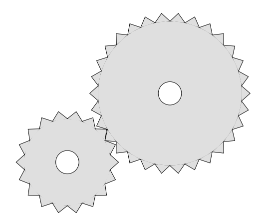


$$x + y = 7$$
$$x^2 + y^2 = 25$$

So, $y = 28 - x$, and $x^2 + (7-x)^2 = 25$. Solving, we get $(x, y) = (4, 3)$.

The vertex of the small square that is the farthest from the vertex A is B. Using the Pythagorean Theorem: $AB = \sqrt{4^2 + 7^2} = \sqrt{65}$.

16. The larger gear has 25 teeth, and the smaller has 15 teeth. If the larger gear rotates through an angle of 90°, through what angle measure does the small gear rotate?



(a) 54° **(b)** 150° **(c)** 90° **(d)** 180°

(e) None of the above

Solution. **B**

On the larger gear, there is 25 "teeth" in 360°. So in 90°, there will be $25/4 = 6\frac{1}{4}$ teeth. On the smaller gear, each "tooth" corresponds to the angle of $360°/15 = 24°$. So $6\frac{1}{4}$ teeth will correspond to the rotation of $6\frac{1}{4} \cdot 24° = 150°$.

17. The Beatles (Paul, John, George and Ringo) took a math exam. Paul got correct half of the questions plus 7 questions, John got correct one third of the questions plus 17 questions, George got correct one fourth of the questions plus 22 questions, and Ringo got correct one fifth of the questions plus 25 questions. There were between one and 100 questions on the exam and each Beatle got an integer number of questions correct. Which Beatle got the most questions correct?

(a) Only Paul **(b)** Only John **(c)** Only George **(d)** Only Ringo

(e) At least two Beatles got the most questions correct.

Solution. **E**

Since each beatle answered an integer number of problems correctly, the total numeber of questions must be divisible by 2, 3, 4 and 5. The only number between 1 and 100 that is divisible by all these numbers is 60. So there was 60 questions. Writing down how many questions each Beatle got right, we have

Paul: $60/2 + 7 = 37$ questions. **John:** $60/3 + 17 = 37$ questions.

George: $60/4 + 22 = 37$ questions. **Ringo:** $60/5 + 25 = 37$ questions.

18. If L is the line whose equation is $ax + by = c$. Let M be the reflection of L through the y-axis, and let N be the reflection of L through the x-axis. Which of the following must be true about M and N for all choices of a, b, and c.

(a) The x-intercepts of M and N are equal

(b) The y-intercepts of M and N are equal

(c) The slopes of M and N are equal

(d) The slopes of M and N are reciprocal

(e) None of the above

SOLUTION. **C**

Reflecting a graph over the x-axis results in the line M whose equation is $ax - by = c$, while a reflection through the y-axis results in the line N whose equation is $-ax + by = c$. Both clearly have slope equal to $\frac{a}{b}$ (from, say, the slope-intercept form of the equation).

19. Let $f(x)$ be a cubic polynomial in one variable x such that $f(0) \neq 0$ and with no repeated roots. Suppose that $f(x)$ and $f(-x)$ share a common root. Which of the following polynomials qualifies to be the least common multiple of $f(x)$ and $f(-x)$? (The numbers a, b and c below are distinct)

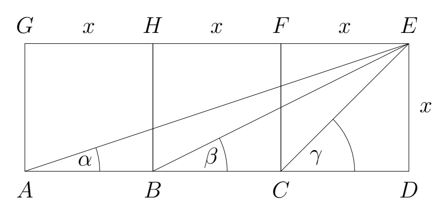
(a) $(x^2 - a)^2(x - b)(x - c)$ **(b)** $(x^2 - a)^2(x^2 - b)$ **(c)** $(x^2 - a)(x - b)(x - c)$

(d) $(x^2 - a)(x^2 - b)$ **(e)** None of the above

SOLUTION. **D**

Let a be the common root of $f(x)$ and $f(-x)$. Then $a \neq 0$ and $f(-a) = 0$, so $-a$ is also a root of $f(x)$. There must be one more root of $f(x)$, we can call it b (a cubic polynomial cannot have exactly two simple roots). So the factors of $f(x)$ are $x - a$, $x + a$ and $x - b$. The factors of $f(-x)$ will then be $x + a$, $x - a$ and $x + b$. So the least common multiple could be $(x - a)(x + a)(x - b)(x + b)$, choice **D**.

20. In the figure to the right, $ABGH$, $BCFG$ and $CDEF$ are congruent squares with side length x. The sum of the angles $\alpha + \beta$ equals



(a) $\pi/3$ **(b)** 3α **(c)** γ **(d)** $\pi/2$

(e) None of the above

SOLUTION. **C**

From the right triangles ADE, BDE, and CDE, respectively, we get $\tan\alpha = 1/3$ and $\tan\beta = 1/2$. The addition formula for tangent is

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \tan(\alpha + \beta) = \frac{\frac{1}{3} + \frac{1}{2}}{1 - \frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

Since the angle $\alpha + \beta$ is in the first quadrant, it has to be $\pi/4 (= \gamma)$.

21. The three little pigs are digging a moat to keep nasty wolves away. The first two pigs, working together, could dig the moat in two hours. The first and third pigs, working together, could dig the moat in one hour and twelve minutes. The second and third pigs, working together, could complete the job in an hour and a half. How long will it take all three pigs working together to dig the moat?

    (a) 36 min **(b)** 45 min. **(c)** 50 min. **(d)** 54 min. **(e)** One hour.

    SOLUTION. **E**

    Let x be the time (in minutes) it takes the first pig to dig the moat alone, y the time (in minutes) it takes the second pig to dig the moat alone, and z the time (in minutes) it takes the third pig. Considering the fraction of the job each pair of pigs can complete in one minute, we get the equations

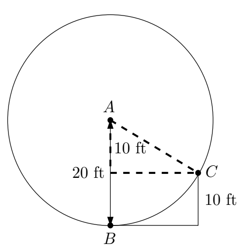
$$\frac{1}{x}+\frac{1}{y}=\frac{1}{120}, \qquad \frac{1}{x}+\frac{1}{z}=\frac{1}{72}, \qquad \frac{1}{y}+\frac{1}{z}=\frac{1}{90}.$$

    Adding the three equations, we get $\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{120}+\frac{1}{72}+\frac{1}{90}=\frac{3+5+4}{360}=\frac{1}{30}$. Hence $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{60}$. This is the fraction of the moat all three pigs working together would dig in one minute. Therefore it would take them 60 minutes to finish the whole moat.

22. Riders on a Ferris wheel travel in a circle in a vertical plane. A particular wheel has radius 20 ft and rotates at a constant rate of one revolution per minute. How many seconds does it take a rider travel from the bottom of the wheel to a point 10 vertical feet above the bottom?

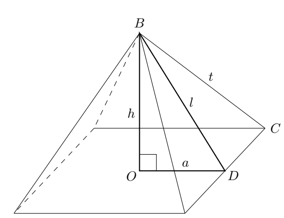
    (a) 5 **(b)** 6 **(c)** 7.5 **(d)** 10 **(e)** None of the above

    SOLUTION. **D**



    The dashed triangle is a right triangle with vertical leg 10 ft and hypotenuse 20 ft. The cosine $\angle BAC$ is $10/20 = 1/2$; hence, the $m\angle BAC = \pi/3$. In other words, the angle is $1/6$ of the full revolution. Since one full revolution takes 60 seconds, and the rate of rotation is constant, it takes 10 seconds to get from the bottom (point B) to the point C.

23. For the regular pyramid to the right, the ratio $h : a$ is $\sqrt{3} : 1$. Find the ratio $t : a$



    (a) $2 : 1$ **(b)** $2\sqrt{2} : 1$ **(c)** $4 : 1$ **(d)** $2\sqrt{5} : 1$

    (e) None of the above

    SOLUTION. **D**

    Note that the side of the base equals $2a$. Hence $EB = a$. By hypothesis, $h = \sqrt{3}a$, hence in the right triangle POE, $PE = \sqrt{h^2 + a^2} = \sqrt{3a^2 + a^2} = 2a$. Since in $\triangle APB$, $PA = PB$, and since E is the mid-point of AB, the $\triangle PEB$ is a right triagle. Hence $PB = \sqrt{(PE)^2 + (EB)^2} = \sqrt{4a^2 + a^2} = \sqrt{5}a$

24. Roadrunner and Coyote are running on a highway simultaneously starting at the same point. Each maintains a constant speed within each 4-mile interval of the road. The speed of Coyote doubles from one interval to the next, while the speed of Roadrunner is reduced by half after

each interval. Roadrunner starts at the speed of 32 mph and Coyote at 1 mph. By the time Coyote catches up with Roadrunner, how many (complete) miles would each have run?

(a) 20 **(b)** 22 **(c)** 24 **(d)** 26 **(e)** None of the above

SOLUTION. **C**

During the k-th 4-mile segment, Cayote's speed is 2^{k-1} and Roadrunner's speed is $32/2^{k-1} = 2^{6-k}$. So it take Cayote $4 \div 2^{k-1} = 2^{3-k}$ hours to finish the k-th segment, while it takes Roadrunner $4 \div 2^{6-k} = 2^{k-4}$ hours. The table below shows the times to complete the successive 4-mile segments

Segment k	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
Cayote's time	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
RR's time	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

From the table it is clear that the y both complete 6 full segments is the same amount of time $t = 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$.

Remark: On could formaly compute time it takes each to complete a given distance d. First writing $d = 4q + r$ with $0 \le r < 4$; q being the complete number of 4-mile segment in d. From the speed of Cayote in each segment, it takes Cayote

$$\begin{aligned} t_C &= \left(\sum_{k=1}^{q} (\text{time to finish the } k\text{-th segment})\right) + r/(\text{speed during the } (q+1)\text{-st segment}) \\ &= \sum_{k=1}^{q} 2^{3-k} + \frac{r}{2^q} = 4\left(\frac{(\frac{1}{2})^q - 1}{\frac{1}{2} - 1}\right) + \frac{r}{2^q} = 2^{-q}\left(8(2^q - 1) + r\right). \end{aligned}$$

where we used the sum formula of a geometric series. Computing the time t_R for Roadrunner in a similar way, then equating t_C and t_R, will get you $q = 6$; but this is more time consuming than the (somewhat ad hoc) solution we presented.

25. The vertex of the parabola that passes through the points $(-1, 0)$, $(0, 1)$ and $(1, 3)$ is

(a) $\left(-\frac{3}{2}, -\frac{1}{8}\right)$ **(b)** $\left(\frac{3}{2}, 8\right)$ **(c)** $(-1, -1)$

(d) $\left(\frac{1}{2}, \frac{3}{2}\right)$ **(e)** None of the above

SOLUTION. **A**

Write $y = ax^2 + bx + c$. Setting $(x, y) = (0, 1)$ implies $c = 1$. To find a and b, we plug in the remaining two points and get:

$$\begin{aligned} 0 &= a - b + 1 \\ 3 &= a + b + 1 \end{aligned}$$

Solving the system, we get $a = 1/2$ and $b = 3/2$. So the parabola is $y = \frac{1}{2}x^2 + \frac{3}{2}x + 1$ The x-coordinate of the vertex is $x = \frac{-b}{2a} = -\frac{3}{2}$. Putting this into the equation, we get the y-coordinate $y = -\frac{1}{8}$.