These are examples of solutions to the problems on the Level II exam. There are certainly different ways to do each problem.

1) $\left(\frac{1}{4}\right)^{-\frac{1}{4}} =$

a. $\sqrt{2}$ b. $-\sqrt{2}$ c.16 d. - 16 e. None of the above

Solution (A): $\left(\frac{1}{4}\right)^{-\frac{1}{4}} = 4^{1/4} = (2^2)^{1/4} = 2^{1/2} = \sqrt{2}$

2) Let f be the function defined by $f(x) = ax^2 - \sqrt{2}$ for some positive a. If $f(f(\sqrt{2})) = -\sqrt{2}$, then a is

a. $2-\sqrt{2}$ b. $\frac{\sqrt{2}}{2}$ c. $\frac{2+\sqrt{2}}{2}$ d. $\frac{2-\sqrt{2}}{2}$ e. None of the above

Solution (B): If $f(x) = -\sqrt{2}$, then x must be 0. Therefore, $f(\sqrt{2}) = 0$. Solving $f(\sqrt{2}) = 2a - \sqrt{2} = 0$, we get $a = \frac{\sqrt{2}}{2}$.

3) As a promotional campaign a music store gave away CDs. Four people received CDs. The first person got 1/3 of the CDs, the second person got 1/4 of the CDs, the third person got 1/6 of the CDs and the fourth person got 12 CDs. How many CDs did the first person get?

a. 8 b. 10 c. 14 d. 16 e. None of the above

Solution (D): If T is the total number of CDs, then $\frac{1}{3}T + \frac{1}{4}T + \frac{1}{6}T + 12 = T$. Multiplying by 12, we have $4T + 3T + 2T + 144 = 12T$, and so $3T = 144$ or $T = 48$. The first person received 1/3 or 16 CDs.

4) Let f be a degree 3 polynomial with leading coefficient 1. Suppose that $x = 3$ is a double zero of f and that $f(0) = 3$. The third zero of f is

a. -2 b. –1 c. – 1/2 d. -1/3 e. None of the above

Solution (D): Since 3 is a double zero, $f(x) = (x-3)^2(x-a)$, where a is the third zero. Therefore, $f(0) = -9a = 3$ and $a = -\frac{1}{3}$.

5) Four whole numbers, when added three at a time, give the sums 180, 197, 208 and 222. What is the largest of the four numbers?

a. 77 b. 83 c. 89 d. 95 e. None of the above

Solution (C): Let the four numbers be w, x, y, z with $w \le x \le y \le z$. Then,

$$\begin{cases} w + x + y \quad\quad = 180 \\ w + x \quad\; + z = 197 \\ w + \quad y + z = 208 \\ \quad x + y + z = 222 \end{cases}$$

and adding the four equations we get $3(w + x + y + z) = 807$ or

$w + x + y + z = 269$. Since $w + x + y = 180$, $z = 269 - 180 = 89$.

6) If $f(x) = 1 - x^2$, find a constant c so that $\dfrac{f(a+h) - f(a)}{h} = c(2a + h)$.

a. c = 1 b. c = -1 c. c = 2 d. c = -2 e. None of the above

Solution (B): $f(a+h) = 1 - (a+h)^2 = 1 - a^2 - 2ah - h^2$ and $f(a) = 1 - a^2$, so $\dfrac{f(a+h) - f(a)}{h} = \dfrac{-2ah - h^2}{h} = -2a - h$. Now, $-2a - h = c(2a + h)$ implies $c = -1$.

7) Which of the following equations have the same graph?

I. $y = x - 2$ II. $y = \dfrac{x^2 - 4}{x + 2}$ III. $(x + 2)y = x^2 - 4$

a. I and II only b. I and III only c. II and III only d. I, II, and III
e. None. All the equations have different graphs.

Solution (E): None can have the same graph. (I) and (III) are defined for all x, but when $x = -2$, (I) has a y-value of –4 and y can be any real number in (III). That is, the graph of (III) is the line $y = x - 2$ plus the vertical line $x = -2$. Neither (I) or (III) can be the same as (II) since (II) is not defined at $x = -2$. That is, the graph of (II) is the line $y = x - 2$ with the point (-2,-4) removed.

8) A tank has two identical drainage valves. With only one valve open, the full tank can be drained in 4 hours. If we start draining the full tank using both valves until the tank is ¼ full, then let the rest drain through one valve only, the time it will take to drain the tank is

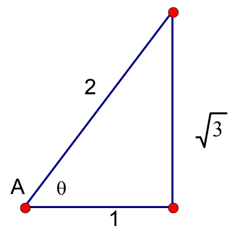
a. 2.25 hours b. 2.5 hours c. 2.75 hours d. 3 hours e. None of the above

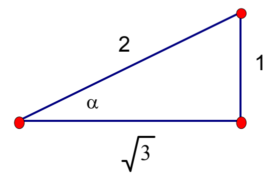
Solution (B): One valve drains at a rate of ¼ tank/hr, so two identical valves will drain at a rate of ½ tank/hr. Let t be the time when both valves are open (to drain ¾ of the tank),then $\frac{1}{2}t = \frac{3}{4}$ or $t = 1.5$ hours. The remaining ¼ of the tank can be drained in 1 hour with one valve, so the total time is 2.5 hours.

9) Find the integer a so that $\tan\left[\sin^{-1}(\frac{\sqrt{3}}{2})\right]+\tan\left[\cos^{-1}(\frac{\sqrt{3}}{2})\right]=\frac{a}{\sqrt{3}}$

a. 1 b. 2 c. 3 d. 4 e. None of the above

Solution (D): Now $\sin^{-1}\left(\sqrt{3}/2\right)$ is the first quadrant angle θ such that $\sin\theta=\sqrt{3}/2$, and similarly $\cos^{-1}\left(\sqrt{3}/2\right)$ is the first quadrant angle α such that $\cos\alpha=\sqrt{3}/2$. If memory of these common angles ($\theta=60^\circ, \alpha=30^\circ$) and the value of the tangent function at each does not prevail, then the right triangle ratios and Pythagorean theorem give $\tan\theta=\sqrt{3}=3/\sqrt{3}$ and $\tan\alpha=1/\sqrt{3}$. Their sum is $4/\sqrt{3}$.





10) If $0\le x\le\frac{\pi}{2}$ and $\sin x=0.1$, then $\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2=$

a. 0.1 b. 0.4 c. 0.7 d. 0.9 e. None of the above

Solution (D): $\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2=\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}+\cos^2\frac{x}{2}=1-2\sin\frac{x}{2}\cos\frac{x}{2}$. But $2\sin\theta\cos\theta=\sin 2\theta$, so $2\sin\frac{x}{2}\cos\frac{x}{2}=\sin x$. Therefore, we have $1-0.1=0.9$.

11) Which of the following trigonometric equations is false for all x.

a. $\sin x=\frac{2}{\sqrt{5}}$ b. tan x = -100 c. $\sec x=\frac{\sqrt{3}}{4}$ d. $\tan^2 x+1=\sec^2 x$ e. None of the above.

Solution (C): The range of the sine function is [-1,1], the tangent function is $(-\infty,\infty)$, and the secant function $(-\infty,-1]\cup[1,\infty)$. Therefore, (a) and (b) are each true for some x and (c) cannot be true. The equation in (d) is true for all x in the domain.

12) When $\pi < t < \frac{3\pi}{2}$, the value of the expression $\frac{1}{1-\cos t} - \frac{1}{1+\cos t}$ in terms of $\sin t$ equals

a. $\frac{-2(1-\sin^2 t)}{\sin^2 t}$. b. $-\frac{2\sqrt{1-\sin^2 t}}{\sin^2 t}$. c. $\frac{2\sqrt{1-\sin^2 t}}{\sin^2 t}$. d. $\frac{2(1-\sin^2 t)}{\sin^2 t}$.
e. None of the above.

Solution (B): The common denominator is $(1-\cos t)(1+\cos t) = 1-\cos^2 t = \sin^2 t$, so we have $\frac{1}{1-\cos t} - \frac{1}{1+\cos t} = \frac{1+\cos t-(1-\cos t)}{(1-\cos t)(1+\cos t)} = \frac{2\cos t}{\sin^2 t}$. Now, $\cos^2 t = 1-\sin^2 t$ or $\cos t = \pm\sqrt{1-\sin^2 t}$. Since $\pi < t < \frac{3\pi}{2}$, $\cos t < 0$, so $\cos t = -\sqrt{1-\sin^2 t}$. This gives the answer in (b).

13) The equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(4,3)$ is

a. $3x+4y=24$ b. $3x-4y=0$ c. $4x-3y=7$ d. $4x+3y=25$ e. None of the above

Solution (D): The radius of a circle is perpendicular to the tangent line at the point of intersection. This circle is centered at (0,0), so the line through (0,0) and (4,3) has slope $3/4$ and equation $y = 3/4\, x$. The tangent line, then, has slope $-4/3$ and goes through (4,3). Its equation is $y-3 = -4/3\,(x-4)$, which simplifies to $4x+3y=25$.

14) How many positive integers less than 50 have an odd number of positive integer divisors?

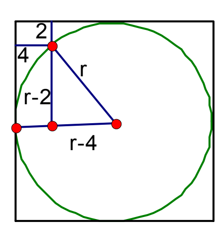
a. 3 b. 5 c. 7 d. 9 e. None of the above

Solution (C): Divisors occur in pairs generally. For example, the divisors of 24 are 1 & 24, 2 & 12, 3 & 8, 4 & 6. In general, if N is a positive integer and d is a divisor, then N/d is also a divisor. The only way to get an odd number of divisors, then, is if $d = N/d$ or $N = d^2$ for some divisor d. Therefore, the positive integers less than 50 with an odd number of divisors are those that are perfect squares. These are $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2$.

15) A circle is inscribed in a square. A 2×4 rectangle shares an upper left corner with the square with its lower right corner on the circle as in the figure. What is the radius of the circle?

a. 4 b. 8 c. 10 d. 12 e. None of the above

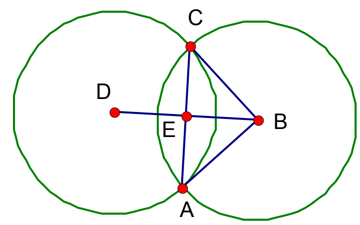
Solution (C): Using the figure, the Pythagorean theorem gives $r^2 = (r-2)^2 + (r-4)^2 = 2r^2 - 12r + 20$, and so $r^2 - 12r + 20 = (r-10)(r-2) = 0$. Since $r \ge 4$, we must have $r = 10$.



16) Two circles have the same radius r. Their centers are a distance of $r\sqrt{2}$ apart. The area of the intersection of the circles is

a. $\frac{\pi r^2}{4}$ b. $\left(2-\frac{\pi}{2}\right)r^2$ c. $\left(\frac{\pi}{2}-1\right)r^2$ d. $\frac{2\pi r^2}{3}$ e. None of the above

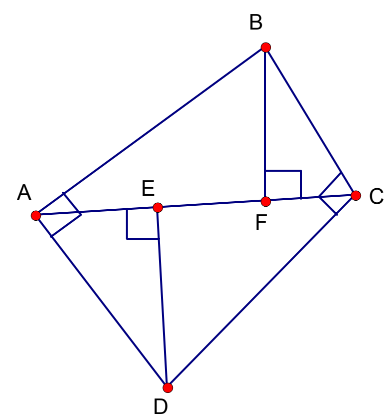
Solution (C): Since $\overline{AC} \perp \overline{BD}$, $BC = r$, and $BE = \frac{\sqrt{2}}{2}r$, ΔBCE is a 45-45-90 and $\angle CBA = 90^\circ$. Therefore, the area of the sector in circle B bounded by the 90 degree arc $\overset{\frown}{AC}$ is $\frac{1}{4}\pi r^2$. The area of triangle ΔABC is $\frac{1}{2}r^2$, so the area of the intersection of the circles is $2\cdot\left(\frac{1}{4}\pi r^2 - \frac{1}{2}r^2\right) = r^2\left(\frac{\pi}{2}-1\right)$.



17) In the figure, ABCD is a quadrilateral with right angles at A and C. Points E and F are on $\overline{AC}$, and $\overline{DE}$ and $\overline{BF}$ are perpendicular to $\overline{AC}$. If AE = 3, DE = 5 and CE = 7, then BF =

a. 3.6 b. 4 c. 4.2 d. 4.5 e. None of the above

Solution (C): $\angle EAD$ and $\angle ABF$ are both complementary to $\angle BAF$, so they are congruent to each other. This makes $\Delta EAD \sim \Delta FBA$ and so $\frac{AE}{BF} = \frac{DE}{AF}$ or $\frac{3}{BF} = \frac{5}{AF} = \frac{5}{3+EF}$. Likewise, $\angle FBC$ and $\angle ECD$ are both complementary to $\angle BCF$, so they are congruent to each other. This makes $\Delta FBC \sim \Delta ECD$ and so $\frac{BF}{EC} = \frac{FC}{DE}$ or $\frac{BF}{7} = \frac{FC}{5} = \frac{7-EF}{5}$. So, by cross multiplying we have $9 + 3EF = 5BF$ and $5BF = 49 - 7EF$. Therefore, $9 + 3EF = 49 - 7EF$ or $EF = 4$. So, $BF = \frac{21}{5} = 4.2$.



18) If $\log_a 2 = 3$ and $\log_2 x = -\frac{3}{2}$, then $\log_a\left(8\cdot\sqrt{\frac{x}{a}}\right)$ is

a. 6.25 b. 8 c. 8.25 d. 9.25 e. None of the above

Solution (A): Using properties of the log function, $\log_a 8\sqrt{\frac{x}{a}} = \log_a 8 + \frac{1}{2}(\log_a x - \log_a a)$. Now

$\log_a a = 1$ and $\log_a 8 = \log_a 2^3 = 3\log_a 2 = 3\cdot 3 = 9$. Also, $\log_a x = \frac{\log_2 x}{\log_2 a} = \frac{-3/2}{1/3} = \frac{-9}{2}$ since

$\log_2 a = \frac{\log_a a}{\log_a 2} = \frac{1}{3}$. Therefore, $\log_a 8\sqrt{\frac{x}{a}} = \log_a 8 + \frac{1}{2}(\log_a x - \log_a a) = 9 + \frac{1}{2}\left(\frac{-9}{2} - 1\right) = 6.25$.

19) If $\log_4 2a^{11} = 6$, then a equals

a. 2 b. 3 c. $\sqrt{2}$ d. –2 e. None of the above

Solution (A): $\log_4 2a^{11} = 6$ if and only if $4^6 = 2^{12} = 2a^{11}$. Therefore, $a = 2$.

20) In how many ways can 6 people be lined up to get on a bus if 2 specific persons refuse to follow each other?

a. 240 b. 480 c. 600 d. 720 e. None of the above

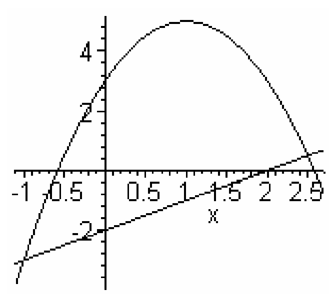
Solution (B): There are 10 positions in line these two people could take. If positions are numbered 1 through 6, they could be 1,3; 1,4; 1,5; 1,6; 2,4; 2,5; 2,6; 3,5; 3,6; 4,6 since they cannot be together. For each of these 10 positions, there are 4! orderings of the other 4 people and 2! orderings of the 2 people. Therefore, there are a total of $10 \cdot 4! \cdot 2! = 480$ orderings.

21) Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta, and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can Professor Alpha, Beta, and Gamma choose their chairs?

a. 12 b. 36 c. 60 d. 84 e. None of the above

Solution (C): A professor cannot be on the end, so they can only be arranged in the middle seven chairs. If we number these 7 chairs 1 through 7, the professors can be in one of the 10 positions listed: (1,3,5), (1,3,6), (1,3,7), (1,4,6), (1,4,7), (1,5,7), (2,4,6), (2,4,7), (2,5,7), (3,5,7). For each of these 10, the professors can be ordered in any of 3! = 6 ways. The total, then is (10)(6) = 60 ways.

22) Find the maximum vertical distance d between the parabola $y = -2x^2 + 4x + 3$ and the line $y = x - 2$ throughout the bounded region in the figure.



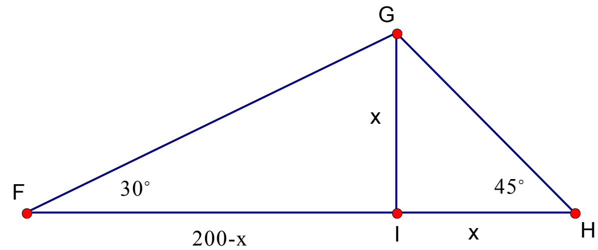
a. $\frac{47}{8}$ b. $\frac{49}{8}$ c. $\frac{50}{8}$ d. $\frac{48}{8}$ e. None of the above.

Solution (B): The vertical distance is given by $-2x^2 + 4x + 3 - (x - 2) = -2x^2 + 3x + 5$, which is a parabola opening down. Its maximum value is the y-coordinate of the vertex which has x-coordinate $\frac{-b}{2a} = \frac{-3}{2(-2)} = \frac{3}{4}$. The y – coordinate, then, is $-2\left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right) + 5 = \frac{-9}{8} + \frac{18}{8} + \frac{40}{8} = \frac{49}{8}$.

23) Two men stand 200 feet apart with a vertical flagpole in between them. If the angles of elevation of the top of the flagpole are $30°$ and $45°$, how tall is the pole?

a. $\frac{200}{1+\sqrt{3}}$ b. $\frac{200\sqrt{3}}{1+\sqrt{3}}$ c. $\frac{200\sqrt{3}}{\sqrt{3}-1}$ d. $\frac{200}{\sqrt{3}-1}$ e. None of the above

Solution (A): Let $HI = x$, then $FI = 200 - x$. Since ΔGHI is a 45-45-90 triangle, the height of the flagpole is $GI = x$ as well. Since ΔFGI is a 30-60-90 triangle, $\frac{FI}{GI} = \sqrt{3}$ or $\frac{200-x}{x} = \sqrt{3}$. Solving for x, $200 - x = \sqrt{3}x$ or $x = \frac{200}{1+\sqrt{3}}$.



24) At noon, a car and a van are 120 miles apart on a straight road and driving toward each other at a speed of 40 mi/hr. A fly starts from the front bumper of the van at noon and flies to the bumper of the car, then immediately back to the bumper of the van, back to the car, and so on, until the car and the van meet. If the fly flies at a speed of 100 mi/hr, what is the total distance it travels?

a. 150 miles b. 120 miles c. 100 miles d. 200 miles e. None of the above

Solution (A): Let t be the time until the van and the car meet, which is also the total time that the fly is flying between the vans, then $40t + 40t = 120$ or $t = 1.5$ hours. Since the fly is flying at 100 mi/hr, the total distance is $(1.5)(100) = 150$ miles.

25) An amoeba propagates by simple division; each split into two takes 3 minutes to complete. When such an amoeba is put into a glass container with a nutrient fluid, the container will be full of amoebas in one hour. How long would it take for the container to be filled if we start with not one amoeba, but two?

a. 30 min b. 45 min c. 51 min d. 57 min e. None of the above

Solution (D): If we let n be the number of 3-minute intervals, then beginning with one amoeba, there are 2^n amoebas in the container. There will be 2^{20} amoebas when it is full since it will fill in one hour. If we begin with 2 amoebas, then there will $2 \cdot 2^n = 2^{n+1}$ amoebas in the container after n three-minute intervals. So it will be full when $2^{n+1} = 2^{20}$ or $n = 19$ three-minute intervals. This is 57 minutes.