April 11, 2003 **Level I Exam Solutions** **Math Olympics 2003**

These are examples of solutions to the problems on the Level I exam. There are certainly different ways to do each of the problems.

1) $\sqrt{\frac{1}{9}+\frac{1}{16}}=$

a. $\frac{1}{5}$ b. $\frac{2}{7}$ c. $\frac{5}{12}$ d. $\frac{7}{12}$ e. None of the above

Solution (C): $\frac{1}{9}=\frac{16}{144}$ and $\frac{1}{16}=\frac{9}{144}$, so their sum is $\frac{25}{144}$ and $\sqrt{\frac{25}{144}}=\frac{5}{12}$.

2) Which of these numbers is largest?

a. $\sqrt{\sqrt[3]{5\cdot 6}}$ b. $\sqrt{6\sqrt[3]{5}}$ c. $\sqrt{5\sqrt[3]{6}}$ d. $\sqrt[3]{5\sqrt{6}}$ e. $\sqrt[3]{6\sqrt{5}}$

Solution (B): Raise each to the 6th power. Since $a>b>0$ if and only if $a^6>b^6$ this will not change the order of the values. After raising to the 6th power: a) $5\cdot 6$ b) $6^3\cdot 5$ c) $5^3\cdot 6$ d) $5^2\cdot 6$ e) $6^2\cdot 5$ and (b) is the largest.

3) Last year a bicycle cost \$160 and a cycling helmet cost \$40. This year the cost of the bicycle increased by 5% and the cost of the helmet increased by 10%. The percent increase in the combined cost of the bicycle and the helmet is

a. 6% b. 7% c. 7.5% d. 8% e. None of the above

Solution (A): The bicycle went up to $160+0.05(160)=168$, and the helmet up to $40+0.10(40)=44$, so the total cost increased from 200 to 212, which is an increase of $\frac{12}{200}=\frac{6}{100}$ or 6%.

4) A vacuum pump removes ½ of the air in a container with each stroke. After 5 strokes, the percentage of the original amount of air that remains in the container will be

a. ½ % b. 1/32% c. 3.125% d. 1/8 % e. None of the above

Solution (C): The percentage goes from 100% to 50% to 25% to 12.5% to 6.25% to 3.125 % since it is reduced by ½ each stroke.

5) The ratio of w to x is 4:3, of y to z is 3:2, and of z to x is 1:6. What is the ratio of w to y?

a. 1:3 b. 16:3 c. 20:3 d. 12:1 e. None of the above

Solution (B): $\frac{w}{y}=\frac{w}{x}\cdot\frac{x}{z}\cdot\frac{z}{y}=\frac{4}{3}\cdot\frac{6}{1}\cdot\frac{2}{3}=\frac{16}{3}$

6) Find the difference of $\frac{1}{x+1}$ and $\frac{x-1}{x^2-1}$.

a. No difference because both are undefined at -1. b. $\frac{-2}{x^2-1}$ c. 0

d. $\frac{1}{x-1}$ e. None of the above

Solution (C): $\frac{1}{x-1} - \frac{x+1}{x^2-1} = \frac{x+1}{(x-1)(x+1)} - \frac{x+1}{(x-1)(x+1)} = 0$

7) The number of real solutions of the equation $|x-2|+|x-3|=1$ is

a. 0 b. 1 c. 2 d. 3 e. More than 3

Solution (E): For any $2 \le x \le 3$, $|x-2| = x-2$ and $|x-3| = 3-x$, so $|x-2|+|x-3| = x-2+3-x = 1$. There are infinitely many solutions.

8) The sum of the solutions to $x^2 - x = 6$ is

a. 1 b. –1 c. –5 d. 13 e. None of the above

Solution (A): $x^2 - x - 6 = (x-3)(x+2) = 0$ when $x = 3, -2$ and $3 + (-2) = 1$

9) If $f(x) = 1 - x^2$, find a constant c so that $\frac{f(a+h)-f(a)}{h} = c(2a+h)$.

a. c = 2 b. c = -2 c. c = 1 d. c = -1 e. None of the above.

Solution (D): $f(a+h) = 1-(a+h)^2 = 1-a^2-2ah-h^2$ and $f(a) = 1-a^2$, so $\frac{f(a+h)-f(a)}{h} = \frac{-2ah-h^2}{h} = -2a-h$. Now, $-2a-h = c(2a+h)$ implies $c = -1$.

10) If $m > 0$ and the points $(m,3)$ and $(1,m)$ lie on a line with slope m, then $m =$

a. 1 b. $\sqrt{2}$ c. $\sqrt{3}$ d. 2 e. None of the above

Solution (C): The slope of the line containing $(m,3)$ and $(1,m)$ is $\frac{m-3}{1-m}$. So, $m = \frac{m-3}{1-m}$ or $m - m^2 = m - 3$. Therefore, $m^2 - 3 = 0$ and $m = \sqrt{3}$ is the positive solution.

11) For how many integers n between 1 and 100 does $x^2 + x - n$ factor into the product of two linear factors with integer coefficients.

a. 8 b. 9 c.10 d.12 e. None of the above

Solution (B) : If $x^2 + x - n = (x+a)(x-b)$ with $a, b > 0$, then $a - b = 1$ and $ab = n$. Therefore, n is the product of consecutive integers: $1 \cdot 2, 2 \cdot 3, 3 \cdot 4, \cdots, 8 \cdot 9, 9 \cdot 10$ are all the possibilities less than 100, and there are 9.

12) For the triangle formed by the points $A(-3,2), B(5,4),$ and $C(3,-8)$, write the equation of the line that contains the altitude of the triangle through point C in the form $y = mx + b$.

a. $y = \frac{1}{4}x + 4$ b. $y = -\frac{1}{4}x - 7\frac{1}{4}$ c. $y = -4x + 4$ d. $y = -4x + 3$

e. None of the above

Solution (C) : The line that contains the altitude is the line through C that is perpendicular to the line $\overleftrightarrow{AB}$. The slope of $\overleftrightarrow{AB}$ is $\frac{4-2}{5-(-3)} = \frac{1}{4}$, so the line containing the altitude has slope -4. The point-slope equation of the line is $y + 8 = -4(x - 3)$, which gives $y = -4x + 4$.

13) A parent is currently 3 times as old as his/her child; and in 10 years he/she will be twice as old as his/her child. How many years older is the parent than the child now.

a. 15 years b. 25 years c. 35 years d. 45 years e. None of the above

Solution (E): If P is the parents current age and C is the child's current age, then $P = 3C$. In 10 years, they will be $P + 10$ and $C + 10$ years old, respectively. Solving $P + 10 = 2(C + 10)$, we get $P = 2C + 10$ and currently $P = 3C$, so $3C = 2C + 10$ or $C = 10$. Therefore, $C = 10$ and $P = 30$ which gives a difference of 20 years.

14) The following system of equations has only one solution if

$$\begin{cases} kx + y = 1 \\ x + ky = 1 \end{cases}$$

a. k = 1 b. k = 0 c. $k \geq 0$ d. $k \neq \pm 1$ e. None of the above

Solution (D): If $k = 0$, then (1,1) is a solution. If $k \neq 0$, then multiplying the second equation by $-k$ and adding the first equation, we get $(1 - k^2)x = 1 - k$. If $k = 1$, then $0x = 0$ is true for all x (infinite number of solutions – same line), and if $k = -1$ then $0x = 2$ is never true (no solution – distinct parallel lines). If $k \neq \pm 1$, then we can divide by $1 - k^2$ and get the unique value of $x = \frac{1-k}{1-k^2} = \frac{1}{1+k}$ (i.e. just one solution).

15) The concentration of a mixture consisting of 10 gallons 20%-acid and 40 gallons 15%-acid is

a. 12%-acid. b. 16%-acid. c. 17%-acid. d. 15% acid. e. None of the above.

Solution (B): Let r be the % acid in the mixture, then $20(10)+15(40)=r(50)$. So, $r=\frac{800}{50}=16$.

16) What is the average of the two solutions of the arbitrary quadratic equation $ax^2+bx+c=0$?

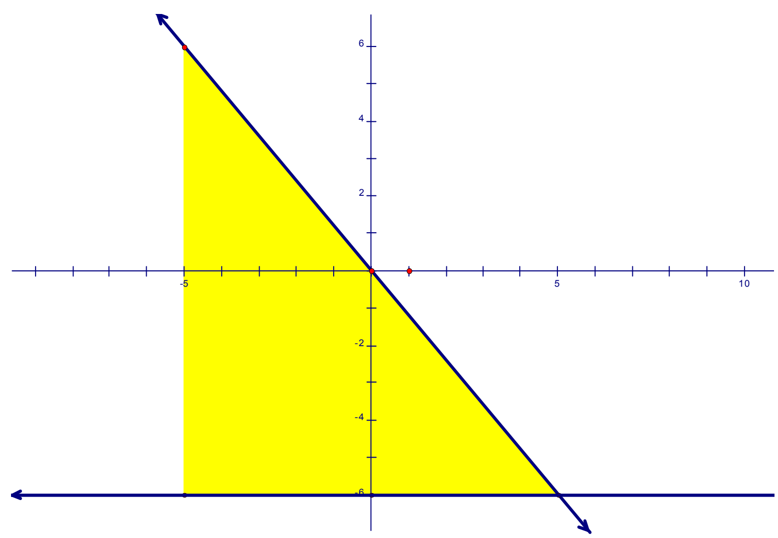
a. $\frac{b}{2a}$ b. $-\frac{b}{2a}$ c. $-\frac{c}{2a}$ d. $-\frac{b}{a}$ e. None of the above.

Solution (B): From the quadratic formula, the two solutions are $x=\frac{-b+\sqrt{b^2-4ac}}{2a}$, $x=\frac{-b-\sqrt{b^2-4ac}}{2a}$. The sum of these two solutions is $\frac{-2b}{2a}=\frac{-b}{a}$, so the average is $\frac{-b}{2a}$.

17) A line passes through (5,-6). Which of the following are possible values of the slope m of the line, if the line never enters the first quadrant?

a. $m\le\frac{-6}{5}$ b. $m\le\frac{-5}{6}$ c. $\frac{-5}{6}\le m\le 0$ d. $\frac{-6}{5}\le m\le 0$ e. None of the above

Solution (D): The lines through (5,-6) That do not enter the first quadrant are all the lines in the shaded region shown. The horizontal line has slope 0 and the line through the origin has slope $\frac{-6}{5}$.

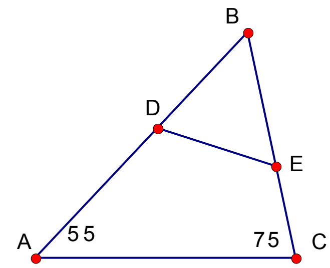


18) Bill scores 78 on a test that had 4 problems worth 7 points each and 24 multiple-choice questions worth 3 points each. If he had one of the 7-point problems wrong, how many of the multiple-choice questions did he miss?

a. 3 b. 4 c. 5 d. 6 e. None of the above

Solution (C): Let x be the number of 3-point problems missed. Since there are a total of 100 points on the exam and he missed one 7-point problem, $100=78+7+3x$. Therefore, $3x=15$ and $x=5$.

19) In $\Delta ABC, \angle A = 55^\circ, \angle C = 75^\circ, D$ is on side $\overline{AB}$ and E is on side $\overline{BC}$. If $DB = BE$, then $\angle BED =$



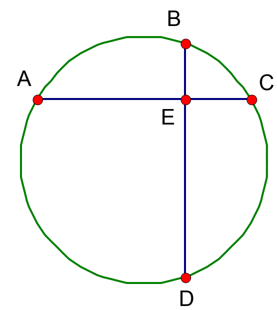
a. 50° b. 55° c. 60° d. 65° e. None of the above

Solution (D): $\angle B = 180 - 55 - 75 = 50^\circ$. Therefore, $\angle BDE + \angle BED = 130^\circ$. Since ΔBDE is isosceles, we know that $\angle BDE = \angle BED = 65^\circ$.

20) If $\angle A$ is four times $\angle B$, and the complement of $\angle B$ is four times the complement of $\angle A$, then $\angle B =$

a. 10° b. 12° c. 15° d. 18° e. None of the above

Solution (D): Let $A = m\angle A, B = m\angle B$. We have $A = 4B$, the complement of A is $90 - A$, and the complement of B is $90 - B$. So, $90 - B = 4(90 - A)$ and substituting $A = 4B$, we have $90 - B = 4(90 - 4B)$. Solving for B, $90 - B = 360 - 16B$ or $15B = 270$, so $B = 18$.



21) If $\overline{AC} \perp \overline{BD}$, DE = 2, BE = 1, and EC = 1 /2, what is the length of AB?

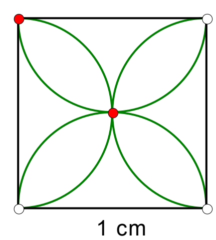
a. $\sqrt{17}$ b. 4 c. $\sqrt{5}$ d. 3 e. None of the above

Solution (A): Since $AE \cdot EC = BE \cdot ED$, $AE \cdot \frac{1}{2} = 2$ or $AE = 4$. Using the Pythagorean Theorem on right triangle ΔABE, we have $AB = \sqrt{4^2 + 1^2} = \sqrt{17}$.

22) The area of the shaded region formed by the four semicircles in the given unit square is

a. $\frac{\pi}{2} - 1$ cm^2 (b) $1 - \frac{\pi}{6}$ cm^2 (c) $1 - \frac{\pi}{8}$ cm^2 (d) $\frac{\pi}{4}$ cm^2 (e) None of the above

Solution (A): If we take the area of the four semicircles, which is $4 \cdot \left(\frac{1}{2} \pi \left(\frac{1}{2} \right)^2 \right) = \frac{\pi}{2}$ square centimeters, this area covers each of the shaded leaves twice and the unshaded areas once. Subtracting the area of the square, which is 1 square centimeter, we are left with just the area of the shaded leaves.



23) In how many ways can 6 people be lined up to get on a bus if 3 specific persons insist on following each other?

a. 144 b. 124 c. 24 d. 720 e. None of the above

Solution (A): The three that insist on following each other could be in any of 4 positions in line (i.e. either first-second-third, second-third-fourth, third-fourth-fifth, or fourth-fifth-sixth, in line). There are $3!$ orderings of these 3 people and $3!$ orderings for the other 3 people. The total number of possible orderings, then, is $4 \cdot 3! \cdot 3! = 144$.

24) Assume that girl-boy births are equally probable. The probability that a family with 5 children has at least one girl is

a. 1/5 b. 31/32 c. 4/5 d. 4/32 e. None of the above

Solution (B): The probability of having no girls (all boys) is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$ since they are independent events, so the probability of having at least one girl is $1 - \frac{1}{32} = \frac{31}{32}$.

25) An old car has to travel a 2-mile route, 1 mile uphill and 1 mile downhill. Because it is so old, the car can climb the first mile – the ascent – no faster than an average speed of 15 mi/hr. How fast does the car need to travel the second mile – on the descent it can go faster, of course – in order to achieve an average speed of 30 mi/hr for the trip?

a. 45 mi/hr b. 60 mi/hr c. 75 mi/hr d. 100 mi/hr e. None of the above

Solution (E): To average 30 mi/hr over the 2 mile trip, the total time would be determined by $\frac{d}{t} = r$, or $\frac{2}{t} = 30$, which gives $t = \frac{1}{15}$ hr. But the least amount of time it could take to go up the one-mile hill is $t = \frac{d}{r} = \frac{1}{15}$ hr. It is impossible to go down the hill in no time.