Euclidean Geometry Review

This is a brief review of Plane Euclidean Geometry - symbols, definitions, and theorems.

Part I: The following are symbols commonly used in geometry:

- $AB$ is the segment from the point A to the point B.
- $AB$ is the measure (length) of the segment $AB$.
- $AB$ is the line containing the points A and B.
- $\overparen{AB}$ is the minor arc of the circle from A to B.
- $\ell{\overparen{AB}}$ is the measure (length) of the arc $\overparen{AB}$.
- $\angle ABC$ is the angle with vertex at B.
- $m\angle ABC$ is the measure of $\angle ABC$.
- $\Delta ABC$ is the triangle with vertices at points A, B and C.
- $\perp$ is the symbol for perpendicular.
- $\parallel$ is the symbol for parallel.
- $\sim$ is the symbol for similar objects.
- $\cong$ is the symbol for congruent objects.
- $\approx$ is the symbol that two numerical quantities are equal.

Symbol Caution: The symbol $\cong$ is used to denote objects are congruent, but this is not the same as the symbol $=$, which refers to two numerical quantities being equal. So, we would not write $AB = CD$ because $AB$ and $CD$ are geometric objects and not numerical quantities. If you wanted to say that they have the same length, then you would write $AB = CD$, since $AB$ and $CD$ refer to the length (numerical quantity) of these. Similarly, be careful in the use of $\angle ABC$ (geometric object) and $m\angle ABC$ (numerical quantity). It is true that

$$AB \cong CD \iff AB = CD$$
$$\angle A \cong \angle B \iff m\angle A = m\angle B$$

But, it is not true that $\overparen{AB} = m\overparen{CD}$ (or $\ell{\overparen{AB}} = \ell\overparen{CD}$) implies $\overparen{AB} \cong \overparen{CD}$.

Part II: Definitions: Below are a few definitions related to lines, angles, triangles, quadrilaterals, polygons and circles. You should very comfortable with these

Lines:
1) Ray: Part of a line that begins at one point on the line and includes all points of the line on one side of the point.
2) Line segment: Part of a line consisting of two points and all points on the line between them.
3) Perpendicular lines: Two lines that intersect to form two right angles.
4) Perpendicular bisector of a line segment: A line (or part of a line) that is both perpendicular to the segment and bisects the segment.

Angles:
5) Vertical angles: A pair of opposite angles formed by two intersecting lines.
6) Adjacent angles: Two angles that have a common vertex and a common side between them.
7) Complementary angles: Two angles whose measures sum to 90 degrees.
8) **Supplementary angles:** Two angles whose measures sum to 180 degrees.
9) **Angle bisector:** A ray that divides the given angle into two smaller, congruent angles.
10) **Interior angle** of a polygon: An angle formed by two sides of the polygon such that the angle lies in the interior of the polygon.
11) **Exterior angle** of a polygon: Any angle that is both supplementary and adjacent to an interior angle of the polygon.
12) When two lines are intersected by a transversal, you should know what **alternate interior angles, alternate exterior angles, corresponding angles, interior angles on the same side, and exterior angles on the same side** are.

**Polygons:**
13) **Polygon:** A plane figure whose sides are line segments that intersect only at their endpoints.
14) **Adjacent sides of a polygon:** Two sides of the polygon that share a vertex.
15) **Equilateral polygon:** A polygon with all sides congruent.
16) **Equiangular polygon:** A polygon with all interior angles congruent.
17) **Regular polygon:** A polygon with all sides and all interior angles congruent.

**Triangles:**
18) **Congruent triangles:** Two triangles whose vertices can be put in one-to-one correspondence such that the three corresponding sides and the three corresponding angles are congruent.
19) **Similar Triangles:** Two triangles whose vertices can be put in one-to-one correspondence such that the three corresponding sides are proportional and the three corresponding angles are congruent.
20) **Isosceles triangle:** A triangle with at least two congruent sides.
21) **Scalene triangle:** A triangle with no two sides congruent.
22) **Median** of a triangle: A segment from a vertex of the triangle to the midpoint of the opposite side.
23) **Altitude** of a triangle: A segment from a vertex of the triangle perpendicular to the opposite side.
24) **Circumcenter** of a triangle: The point of intersection of the three perpendicular bisectors of the sides of the triangle. It is the center of the circumscribed circle to the triangle (see Circles below)
25) **Incenter** of a triangle: The point of intersection of the three angle bisectors of the triangle.
26) **Centroid** of a triangle: The point of intersection of the three medians of the triangle.
27) **Orthocenter** of a triangle: The point of intersection of the three altitudes (extended) of the triangle.

**Quadrilaterals:**
28) **Parallelogram:** a quadrilateral with both pairs of opposite sides parallel.
29) **Rectangle:** a quadrilateral with four right angles
30) **Rhombus:** a quadrilateral with four congruent sides.
31) **Square:** a quadrilateral with four congruent sides and four congruent (and so right) angles.
32) **Trapezoid:** a quadrilateral with exactly one pair of parallel sides.
33) **Isosceles trapezoid:** A trapezoid with the two non-parallel sides congruent.

**Circles:**
34) **Circle** (center, radius): The set of points in the plane that are a fixed distance from a given point in the plane. The fixed distance is the **radius** and the given point is the **center** of the circle.
35) **Chord:** A segment joining two points on the circle.
36) **Diameter:** A chord that contains the center of the circle.
37) **Secant** to a circle: A line that contains exactly two distinct points of the circle.
38) **Tangent** to a circle: A line that contains exactly one point of the circle.
39) **Arc** of a circle (**minor, major**): The part of a circle determined by two points and all the points on the circle between them. If the arc is less than a semi-circle, then it is a **minor arc**. If the arc is more than a semi-circle, then it is a **major arc**.
40) **Central Angle:** An angle whose vertex is the center of the circle.
41) **Inscribed angle:** An angle whose vertex is on the circle and the sides of the angle are chords of the circle.

42) **Intercepted arc:** An angle APB intercepts the arc \( \overparen{AB} \) if A and B are on the circle. \( \overparen{AB} \) is called the **intercepted arc** of angle APB.

43) **The measure of an arc:** The measure of an arc is the measure of the central angle that intercepts the arc.

44) **Sector** of a circle: The region bounded by the sides of a central angle and its intercepted arc.

45) **Inscribed circle** to a polygon: A circle inside a polygon such that each side of the polygon is tangent to the circle.

46) **Circumscribed circle** to a polygon: A circle that contains each vertex of the polygon and each side of the polygon is a chord of the circle.

**Theorems:** Remember that all of the Neutral Theorems are true (some are repeated here). Now, since we are in Euclidean geometry and we are assuming the **Parallel postulate**, we have many more theorems that are consequences of it. Remember that the Parallel postulate is equivalent to **Playfair’s Postulate** – Through any point off a given line, there is a unique line containing the point that is parallel to the given line.

**Lines:**
1) A point P is on the perpendicular bisector of \( \overparen{AB} \) iff \( AP = BP \).

**Angles, Polygons**
2) Vertical angles are congruent.
3) A point P is on the angle bisector of \( \angle ABC \) iff P is equidistant from \( \overparen{BA} \) and \( \overparen{BC} \).
4) Two angles that are complementary or supplementary to the same angle (or congruent angles) are congruent to each other.
5) The sum of all interior angles of a polygon with \( n \) sides is \( (n - 2)180^\circ \).
6) The sum all exterior angles (one at each vertex) of a polygon is \( 360^\circ \).
7) The measure of each interior angle of an \( n \)-sided equiangular polygon is \( \frac{(n - 2)180^\circ}{n} \).
8) Let two lines be intersected by a transversal. The two lines are parallel if and only if any one of the following is true:
   a. A pair of alternate interior angles is congruent
   b. A pair of corresponding angles is congruent
   c. A pair of alternate exterior angles is congruent.
   d. A pair of interior angles on the same side of the transversal are supplementary.
   e. A pair of exterior angles on the same side of the transversal are supplementary.

**Triangles:**
9) **Triangle Congruence Theorems:** SAS, SSS, ASA, AAS
10) The sum of the interior angles of a triangle is \( 180^\circ \).
11) The measure of an exterior angle of a triangle is equal to the sum of the measures of the two non-adjacent interior angles of the triangle.
12) **Triangle Similarity Theorems:** SAS, SSS, AA. Here we mean sides are **proportional** and angles are congruent.
13) A segment from the midpoint of one side of the triangle to the midpoint of a second side is parallel to the third side and it measures half the length of the third side.
14) The altitude to the hypotenuse of a right triangle forms two right triangles that are similar to each other and to the original right triangle.

15) The median to the hypotenuse of a right triangle is one-half the length of the hypotenuse.

16) A line intersecting two sides of a triangle is parallel to the third side if and only if it divides the two sides proportionally.

17) Let \( m \parallel n \parallel p \). Let \( t \) be a transversal intersecting \( m, n, p \) such that the ratio of the lengths of the two segments of \( t \) between the parallels is \( r \). Then, every transversal intersecting \( m, n, p \) will have the ratio of the (corresponding) lengths of the two segments between the parallels equal to \( r \). (This result can be extended to any number of parallel lines).

18) An interior angle bisector of any triangle divides the side of the triangle opposite the angle into segments that are proportional to the other two sides of the triangle.

19) An exterior angle bisector of a triangle determines, with each of the other two vertices of the triangle, segments along the line containing the opposite side of the triangle that are proportional to the two remaining sides of the triangle.

Quadrilaterals:

20) A quadrilateral is a parallelogram if and only if the diagonals bisect each other.

21) Opposite sides of a parallelogram are congruent.

22) Opposite angles of a parallelogram are congruent.

23) A quadrilateral is a rhombus if and only if the diagonals are perpendicular bisectors of each other.

Circles:

24) The measure of an inscribed angle is one-half the measure of its intercepted arc.

25) Corollary to (23): Inscribed angles that intercept the same or congruent arcs are congruent.

26) If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.

27) A chord is perpendicular to the tangent line at one of its endpoints if and only if the chord is a diameter.

28) Chord-Tangent Theorem: The measure of the angle formed by a chord and a tangent line at one endpoint of the chord is one-half its intercepted arc.

29) Two chords are congruent if and only if they have congruent arcs.

30) Two chords are congruent if and only if they are equidistant from the center of the circle.

31) The perpendicular bisector of a chord contains the center of the circle.

32) The line joining the centers of two intersecting circles is the perpendicular bisector of the common chord of the two circles.

33) Two-Chord Theorems: If two chords intersect in the interior of a circle, then
   a. The measure of the angle at the intersection is the average of the measures of the two arcs – one intercepted by the angle and the other intercepted by its vertical angle.
   b. The product of the lengths of the two segments on one chord is equal to the product of the lengths of the two segments on the other chord.
34) **Two-Secant Theorems**: If two secant segments intersect at a point outside a circle, then
   a. The measure of the angle at the point of intersection is half the (positive) difference of the two intercepted arcs.
   b. The product of the length of the whole secant segment and its external part is the same for both secants.

35) **Secant – Tangent Theorems**: If a secant segment and a tangent segment intersect at a point outside a circle, then
   a. The measure of the angle at the point of intersection is half the (positive) difference of the two intercepted arcs.
   b. The product of the length of the whole secant segment and its external part is equal to the square of the length of the tangent segment.

36) **Two-Tangent Theorems**: If two tangent segments intersect at a point outside a circle, then
   a. The measure of the angle at the point of intersection is half the (positive) difference of the two intercepted arcs.
   b. The tangent segments are equal in length (they are congruent).
Euclidean Exercises – Set 1

1) Prove that the perpendicular bisectors of the sides of a triangle intersect at a common point (the circumcenter).
2) Prove that the sum all exterior angles (one at each vertex) of an \( n \)-sided polygon is \( 360^\circ \).
3) **Tessellation or tiling of the plane**: Use polygonal figures to completely cover the plane with no gaps between figures and no overlaps of figures.
   a. **Which regular polygons will tessellate the plane?** Justify your answer. (Here your tessellation will consist of only one type of polygon)
   b. Which **combinations** (more than one) of regular polygons, with each type of polygon having equal side length with all other polygons, can be used to tessellate the plane such that the arrangement of polygons at each vertex is the same? Justify your answer. (There are between 5 and 10 of these)
4) Prove that the measure of an exterior angle of a triangle is equal to the sum of the measures of the two non-adjacent interior angles of the triangle.
5) Prove the median to the hypotenuse of a right triangle is one-half the length of the hypotenuse.
6) Prove that a line intersecting two sides of a triangle is parallel to the third side if and only if it divides the two sides proportionally.
7) Prove that an interior angle bisector of any triangle divides the side of the triangle opposite the angle into segments that are proportional to the other two sides of the triangle.
8) Prove that opposite sides of a parallelogram are congruent.
9) Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
10) Prove that the three medians of a triangle intersect at a common point and that the distance from any vertex to the point of intersection of the medians is two-thirds of the distance from the vertex to the midpoint of the opposite side.

Hint: Begin with triangle ABC and any two medians (they must intersect at a point P). Let AM and BN be the medians, where M is the midpoint of BC and N is the midpoint of AC. Construct 4 lines; one through C, one through B, one through the midpoint of MC, and one through the midpoint of BC, all of which are parallel to AM.
Euclidean Exercises – Set 2

1) Prove that if a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.

2) Prove the Chord – Tangent Theorem. Hint: Begin with the case where the chord is a diameter.

3) Prove the Two-Chord Theorems (both)

4) Prove the converse of the Secant-Tangent (segment length) Theorem: That is: Let $\overline{AEF}$ be a secant segment with $A$ outside the circle and $E, F$ on the circle with $AF > AE$. Prove that if $G$ is any point on the circle satisfying $AG^2 = AF \cdot AE$, then $\overline{AG}$ is tangent to the circle at $G$.

5) In $\triangle ABC$, let $F, G, H$ be on $\overline{BC}, \overline{AC}, \overline{AB}$, respectively such that $\overline{AF}, \overline{BG}, \overline{CH}$ are the three angle bisectors of the interior angles of the triangle. Prove that $\frac{AH \cdot BF \cdot CG}{HB \cdot FC \cdot GA} = 1$.

6) In $\triangle ABC$, let $F, G, H$ be on $\overline{BC}, \overline{AC}, \overline{AB}$, respectively such that $\overline{AF}, \overline{BG}, \overline{CH}$ are the three altitudes of the triangle ($F,G,H$ are the feet of the altitudes – you can assume that the triangle is acute in your picture for the proof so that $F,G,H$ are between the vertices of the triangle. The proof works similarly even if it is obtuse). Prove that $\frac{AH \cdot BF \cdot CG}{HB \cdot FC \cdot GA} = 1$. Hint: Find appropriate similar triangles.

7) Prove that the three altitudes of a triangle intersect at a common point. Hint: Construct a line through each vertex of the triangle that is parallel to the opposite side. This gives you a new larger triangle. How are the altitudes of the original triangle related to the new larger triangle? Parallelograms?

8) In $\triangle ABC$, let $H$ be the orthocenter and $O$ the circumcenter. Let $F$ be the foot of the altitude $\overline{AF}$ (so $H$ is on $\overline{AF}$) (you may assume $F$ is between $B$ and $C$). Let $M$ be the midpoint of $\overline{BC}$. Prove that $AH = 2 \cdot OM$. Hint: Draw the circumcircle ($O$ is the center) and draw diameter $BK$ (so $K$ is on the circle). First compare $OM$ and $CK$. Parallelogram?