Logic Unit

The English language is at the same time both simple and complicated, both vapid and expressive. Consider the three sentences: Buffalo!
   Buffalo buffalo.
   Buffalo buffalo buffalo.
With various inflections, consider how many different meanings could be given to the sentence:
   I didn’t say John stole the book.

In this unit, our goal is to set up a formal language modeling some of the aspects of the English language within which the idea of logical deduction can be examined. Applications can be practical as well as whimsical (for instructive as well as for humorous intent). Without logic, computers would be chaotic. Deep questions of halting and correctness of computer programs involve logic.

In logic a simple statement or proposition is a declarative sentence to which it is meaningful to assign a truth value (either true or false). The following are examples of statements:
   George Washington fought at Waterloo.
   The earth is round like a ball.
   Steve is 6 feet tall. (Here the context would clarify what the truth value would be.)
The following are not statements:
   Open the door.
   Is this class interesting?
   Drive safely.

LOGICAL CONNECTIVES: Logical connectives are used to combine or modify simple statements.
The more common logical connectives are as follows: and, or, negation, conditional, and biconditional. In the development given, p, q, r, etc represent statements.

and (or conjunction)  p and q  notation  p \land q
or (or disjunction)  p or q  notation  p \lor q
not (or negation)  not p  notation  \neg p  (sometimes ~ p is used)
conditional  if p then q  notation  p \rightarrow q  (sometimes a simple arrow is used)
biconditional  p if and only if q  notation  p \leftrightarrow q  (sometimes a simple double arrow is used)

In a conditional if p then q, the p (or whatever is playing the same role) is called the antecedent and q is called the consequent.
The truth values of the compound statements are given in tabular form. Truth tables provide an organized way of examining all possible cases.

<table>
<thead>
<tr>
<th>p</th>
<th>(\neg p)</th>
<th>p</th>
<th>q</th>
<th>p &amp; q</th>
<th>p</th>
<th>q</th>
<th>p \lor q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<td>F</td>
</tr>
</tbody>
</table>

\[ p \quad q \quad p \Rightarrow q \quad p \quad q \quad p \Leftarrow q \]

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p \Rightarrow q</th>
<th>p</th>
<th>q</th>
<th>p \Leftarrow q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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Definition: A **tautology** is a statement which is always true.

In order to make it easy to talk about various truth tables, the standard setup of cases should be used. The standard setup is to have the simple statements listed in alphabetical order and the cases arranged as shown. For tables involving say p, q, r, you need 8 lines or cases. The usual setup is to have the first column to have TTTTFFFF, the second column TFFFTTF, the third column is then TFTFTFTF. For tables involving say p, q, r, s, you need 16 lines or cases. The standard setup is TTTTTTTTTTTTTTTTTTTTFFFF, the second column TTTTFFFFTTTTFFFF, the third TTFFTTFFTTFFTTFF, and the fourth is TFTFTFTFTFTFTF.

Example: Find truth table for \( \neg(p \lor q) \)

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>\neg(p \lor q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
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<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Note the “or” column is completed first and then the negation column is completed.

Exercise: Construct truth table for \(\neg p \land \neg q\). Compare it to the example above. Note the two negation columns will need to be completed before the “and” column can be done.

Exercise: Construct truth table for \(\neg p \lor q\). Note the negation applies only to the p. Compare the results to the column for \(p \Rightarrow q\).

Exercise: In view of the comparison in the previous one, what would you expect to have as main column for \((p \Rightarrow q) \iff (\neg p \lor q)\)? Construct its truth table.
Exercise: Construct truth table for \( p \land (q \lor r) \iff (p \land q) \lor (p \land r) \). Note the “or” on left will need to be done before the “and” on the left can be done. The two “ands” will need to be done on the right before the “or” can be done. The two “ands” will need to be done on the right before the “or” can be done. The biconditional will use the “and” column on the left and the “or” column on the right.

**Properties:**

- **Associative**
  \[
  p \land (q \land r) \iff (p \land q) \land r \\
  p \lor (q \lor r) \iff (p \lor q) \lor r
  \]

- **Distributive**
  \[
  p \land (q \lor r) \iff (p \land q) \lor (p \land r) \\
  p \lor (q \land r) \iff (p \lor q) \land (p \lor r)
  \]

- **DeMorgan Laws**
  \[
  \neg(p \land q) \iff \neg p \land \neg q \\
  \neg(p \lor q) \iff \neg p \lor \neg q
  \]

Exercise: Four men, one of whom was known to have committed a crime, made the following statements when questioned by the police:

- **Archie:** “Dave did it.”
- **Dave:** “Tom did it.”
- **Gus:** “I didn’t do it.”
- **Tom:** “Dave lied when he said I did it.”

a. If only one man is telling the truth, who is the man guilty of the crime?
b. If only one man is lying, who is the man guilty of the crime?

Hint: For each of these, consider four cases. In each part, you can find inherent contradictions in three of the cases. The remaining case will establish who is guilty.

**Quantified Statements:** These are examined here to clarify what is meant when they are used, but even more to examine how their negations are so often mis-constructed.

The two types of quantifiers are **universal** and **existential**.

Examples: universal:
\[
 p \quad \text{All birds are small. It means that of any bird it could be said that “it is small.”}
\]
\[
 \neg p \quad \text{Not all birds are small.}
\]
\[
 \text{Some bird is not small.}
\]
\[
 \text{At least one bird is not small.}
\]
\[
 \text{It is not the case that all birds are small. (This one is not as satisfying as the others).}
\]
A common error is to say that “all birds are not small.” This would mean that there are no small birds. (Don’t tell the hummingbird!)

existential: \( q \)   Some men are tall.
\(~q\)   No man is tall.

All men are not tall.

These will be explored later in the section on arguments.

**Important Related Conditionals**

Given the conditional \( p \Rightarrow q \), three important related conditionals are its *converse, inverse, and contrapositive*.

Its converse is \( q \Rightarrow p \).

Its inverse is \( \sim p \Rightarrow \sim q \).

Its contrapositive is \( \sim q \Rightarrow \sim p \).

Exercise: Compare the truth tables for \( p \Rightarrow q \), \( q \Rightarrow p \), \( \sim p \Rightarrow \sim q \), \( \sim q \Rightarrow \sim p \).

Exercise: Use \( p: \) I study hard.
\( q: \) I will make a good grade.
Express in words:
\( \sim p \)
\( \sim q \)
\( p \Rightarrow q \)
\( q \Rightarrow p \)
\( \sim p \Rightarrow \sim q \)
\( \sim q \Rightarrow \sim p \)

Exercise: For each of the following, assign some letter to each simple statement and symbolize the conditional. Write its converse, inverse, and contrapositive in symbolic form and in verbal form.

a. If today is Friday, then I am happy.

b. If I buy a new car or I buy a used car, then I must buy gasoline.
ARGUMENTS:
An argument consists of a set of statements called premises or hypotheses and a statement called the conclusion. An argument is said to be valid if whenever all the premises are true then so is the conclusion. An argument is not valid if a set of truth values for the simple statements can be found which makes all the premises true and the conclusion false. An example of an argument is: If I study hard, then I will make a good grade.
   I study hard.
   \therefore I will make a good grade.

Another example is If I study hard, then I will make a good grade.
   I don’t study hard.
   \therefore I won’t make a good grade.

The first one could be symbolized as follows: \[ p \implies q \]
   \[
   \begin{array}{c|c}
   p & q \\
   \hline
   T & T \\
   T & F \\
   F & T \\
   F & F \\
   \end{array}
   \]
   \[
   \therefore q
   \]

The second could be symbolized as follow: \[ p \implies q \]
   \[
   \begin{array}{c|c}
   p & q \\
   \hline
   T & T \\
   T & F \\
   F & T \\
   F & F \\
   \end{array}
   \]
   \[
   \neg p
   \]
   \[
   \therefore \neg q
   \]

Three methods of determining and proving validity will be used in this unit.

The first method considered is one which uses truth tables. For a valid argument, the conditional “if (conjunction of premises), then the (conclusion)”, will be a tautology and thus the main column of its truth table will have only T’s. For an argument with several simple statements, this is a time-consuming task. For example,

\[
(p \implies q) \land p \implies q \]

would produce all T’s in its main column(the second \( \implies \)). An invalid argument would produce at least one F in the main column. The other argument when examined this way would produce an F in at least one spot of the main column. For example,

\[
(p \implies q) \land \neg p \implies \neg q \]

would produce an F in its main column(the second \( \implies \)) and hence is an example of an invalid argument.

Another method of proof that an argument is valid is the method of contradiction. To do this, show that if the conclusion is false, then one or more of the premises would have to be false. On the other hand, if the conclusion can be false while all the premises are true, the argument is not valid. The truth values will identify a line in the truth table which would produce an F in the
main column. The set of values of the simple statements would then constitute a counterexample
to the assertion that the conclusion follows. If the conclusion can be false in several different
ways, this method can be laborious. However, if there is only one way the conclusion can be
false, this method can be rather quick.

Some of the following are valid and some are not.

\[
p \Rightarrow q \quad p \Rightarrow q \quad p \Rightarrow q
\]
Example: \(\neg p\) \qquad Example: \(q \Rightarrow r\) \qquad Example: \(\neg q\)
\[
\therefore q \quad \therefore q \quad \therefore \neg p
\]

Still another method to prove validity is to use a set of standard arguments or basic rules of
inference to reduce an argument to a simpler argument. We will use the following set of basic
arguments or rules of inference:

**DOUBLE NEGATION**\[ p \quad \neg \neg p \]
\[ \therefore \neg \neg \neg p \quad \therefore p \]

**IMPLICATION**\[ \neg p \lor q \quad p \Rightarrow q \]
\[ \therefore p \Rightarrow q \quad \therefore \neg p \lor q \]

**SIMPLIFICATION**\[ p \land q \quad p \land q \]
\[ \therefore q \quad \therefore p \]

**MODUS PONENS**\[ p \Rightarrow q \]
\[ \therefore p \]

**MODUS TOLLENS**\[ q \]
\[ \therefore \neg q \]

**CONTRAPOSITION**\[ p \Rightarrow q \]
\[ \therefore \neg q \Rightarrow \neg p \]

**DISJUNCTIVE SYLLOGISM**\[ p \lor q \quad p \lor q \]
\[ \therefore \neg p \quad \therefore \neg q \]
\[ \therefore q \quad \therefore p \]

**HYPOTHETICAL SYLLOGISM**\[ p \Rightarrow q \quad q \Rightarrow r \]
\[ \therefore p \Rightarrow r \]

**DeMORGAN’S LAWS**\[ \neg(p \lor q) \quad \neg(p \land q) \]
\[ \therefore \neg p \land \neg q \quad \therefore \neg p \lor \neg q \]
Example:

A.) If the cup is gold, then it is lighter than water.
If the cup is lighter than water, then Janie can carry it.
∴ If the cup is gold, then Janie can carry it.

B.) If the package is not properly addressed or it is too large, then it will not be accepted by the Post Office. This package is not too large. If Thomas addressed this package, then it is properly addressed.
∴ If Thomas addressed this package, then it will be accepted by the Post Office.

Exercise: Using the basic rules of inference, prove the validity of the following:

A.) 
\[
\begin{align*}
\neg p & \Rightarrow q \\
\neg p \lor q & \\
q \Rightarrow r & \\
\therefore r & \\
\therefore \neg p &
\end{align*}
\]

B.) 
\[
\begin{align*}
\neg r & \Rightarrow \neg u \\
\neg r & \\
\therefore \neg p & \\
\therefore \neg u &
\end{align*}
\]

Exercise: Determine the validity of the following arguments. If the argument is not valid, construct a counterexample case. If it is valid, prove validity using one of the last two methods.

A.) 
\[
\begin{align*}
r & \Rightarrow s \\
p & \Rightarrow q \\
r \lor p & \\
\therefore s \lor q &
\end{align*}
\]

B.) 
\[
\begin{align*}
r & \Rightarrow s \\
p & \Rightarrow q \\
\therefore r \lor s & \\
\therefore q &
\end{align*}
\]

C.) 
\[
\begin{align*}
t & \Rightarrow w \\
\neg t & \Rightarrow \neg r \\
\therefore \neg s & \\
\therefore r \lor s & \\
\therefore w &
\end{align*}
\]

Examples of arguments using quantified statements: (Note: some are not valid.)

Example: All German shepherds are dogs.
All dogs bark.
∴ All German shepherds bark.

Example: All teachers are rich.
All rich people are snobs.
∴ All teachers are snobs.
Example: All teachers are rich. 
Tim is a teacher. 
∴ Tim is rich.

Example: No airline pilots eat carrots. 
Jill does not eat carrots. 
∴ Jill is an airline pilot.

Example: No mathematicians eat licorice. 
All college graduates eat licorice. 
∴ No mathematicians are college graduates.

Note: If a picture(diagram) can be drawn which will satisfy all the premises and yet not satisfy the conclusion, then the argument is invalid. On the other hand, if any picture(diagram) satisfying all the premises must also satisfy the conclusion, then the argument is valid.