Has Chaos Been Explained?
Jeffrey Koperski

ABSTRACT

In his recent book, *Explaining Chaos*, Peter Smith presents a new problem in the foundations of chaos theory. Specifically, he argues that the standard ways of justifying idealizations in mathematical models fail when it comes to the infinite intricacy found in strange attractors. I argue that Smith’s analysis undermines much of the explanatory power of chaos theory. A better approach is developed by drawing analogies from the models found in continuum mechanics.

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If a theory were not simpler than the phenomena it was designed to model, it would serve no purpose.—Clifford Truesdell

1 Introduction

To a remarkable extent, the answer to the title question is yes: Peter Smith [1998] explains much about that part of nonlinear dynamics popularly known as chaos theory. On the other hand, there is one important aspect of chaotic models that he instead explains away. Real systems, he argues, cannot have the central property posited within such models. Understanding this mismatch between model and reality is one of Smith’s chief concerns and is also the focus of this article.

Smith’s *Explaining Chaos* is a wide-ranging introduction to the philosophy of chaos theory. Many of the topics discussed in other, nonphilosophical surveys are included: phase spaces, ordinary differential equations (ODEs),

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fractals and strange attractors, mathematical models and real-world applications, and the (so-called) routes to chaos. More importantly for philosophers, it also covers many of the areas that chaos has touched in the philosophy of science: scientific explanations, determinism-randomness-complexity, predictions, approximate truth, and the definition of chaos. The most interesting and original discussion considers the role that ‘infinitely structured models can have in representing a messy world and explaining natural phenomena’ ([1998], p. 165). Smith argues that chaotic models are, in an important sense, more complex and intricate than the physical systems being modeled. How then can one treat such a model realistically? Indeed, how can such models be useful at all? Smith’s answer to this challenge misses the mark—perhaps the only ‘miss’ evident in the book. In the remainder of this article, I first show why it seems that special justification for the use of chaotic models is needed, and then consider Smith’s proposed solution. Section 3 presents three problems with this solution. Section 4 offers a different view of models containing surplus structure that follows the approach used in continuum mechanics. In the end, I argue that philosophers and scientists should have no qualms about treating chaotic models like any other. The intricacy found in such models is not as peculiar as it might appear and defending this view is not as difficult as Smith takes it to be.

2 The problem of infinite structure

Smith’s argument about the use of nonlinear models starts with a challenge that he first raised several years ago (Smith [1991]). Chaotic models, he shows, contain intrinsic falsehoods. Such claims are no longer radical. Several in-depth analyses of physical idealizations have been in print for some time now.1 But unlike frictionless planes and ideal circuits, Smith argues that the falsehood contained in chaotic models is the very thing that makes them so interesting. Chaotic models misrepresent nature at their core, not merely at the periphery.

Some distinctions are needed before pressing on. Recognizing that the word ‘model’, like ‘law’ and ‘theory’, is ambiguous, Smith introduces some helpful categories. Dynamical systems are sets of real-world objects with identifiable states that change over time. These also seem to include idealized systems. Dynamical equations are the sets of equations governing dynamical systems. Smith’s chief concern is with mathematical models, the abstract geometrical structures that capture the state and behavior of dynamical systems. These include trajectories, attractors, and repellors in phase space.

1 See for example Nancy Cartwright ([1983]), William Wimsatt ([1987]), and Ronald Laymon ([1989]).
Let us now put the claim that chaotic models misrepresent nature more precisely: chaotic mathematical models contain properties that do not correspond to anything in the dynamical systems being modeled. The heart of this misrepresentation is the ‘infinite intricacy’ found in strange attractors, specifically their fractal structure (Smith [1998], p. 39). Unlike coastlines and ferns, strange attractors are true fractals in phase space. Material objects are at best fractal-like: self-similarity within some range. For example, given the molecular nature of matter, the fractal geometry of a mountainside cannot be supported at all scales. Eventually the fractal must give way to discrete atoms. In contrast, phase space is composed of mathematical points. Each point in the space represents a possible state of the system and can be represented by an \( n \)-tuple of real numbers. \( n \) corresponds to the dimension of the phase space. Each point belongs to a trajectory (or ‘orbit’) that represents a possible evolution of the system. A phase space with state trajectories is called a phase portrait. In dissipative systems, the phase portrait often contains attractors: sets of points toward which neighboring trajectories flow. Chaotic evolutions are associated with strange attractors in which trajectories exponentially diverge from their neighbors.\(^2\) Trajectories within the basin of attraction wrap around the attractor as \( t \to \infty \), giving a strange attractor its unbounded fractal geometry. Fine-structure is evident at all scales.

Smith’s charge is that infinitely intricate fractal structure—unique to chaotic attractors\(^3\)—is not realistic. This becomes obvious, he argues, when one considers the values of the physical magnitudes that are supposed to correspond to states of the dynamical system. The points on a trajectory governed by a strange attractor represent states that the model takes to be physically possible. However, many of the values corresponding to these points cannot be had by the dynamical system being modeled:

\[
[M]acroscopic quantities of the type dealt with in paradigm chaotic models \ldots cannot have indefinitely precise real number values. Hence their time evolutions cannot really exemplify infinitely intricate trajectories wrapping round a fractal attractor (Smith [1998], p. 39).
\]

In other words, we know from the underlying physics that the state variables cannot take on certain values. But if the fractal structure of the strange attractor is to be interpreted realistically, the variables must take on such values.

Consider an example from an earlier paper (Smith [1991], p. 256). Let \( T \) be the state variable for the temperature of a gas. Gases, as we know, are molecular and \( T \) depends on the average kinetic energy of its constituent

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\(^2\) See Abraham and Shaw ([1992], p. 121) for a helpful illustration of how this happens.

\(^3\) Although imprecise, I will use ‘strange’, ‘chaotic’, and ‘fractal’ interchangeably when referring to attractors. Strictly speaking, there are strange, non-chaotic attractors as well as chaotic attractors that are not fractals.
molecules. In the kinds of dynamical equations that govern strange attractors (coupled, nonlinear ODEs), $T$ is represented by a real number. Smith shows, however, that it is not physically possible for temperature to have an infinitely precise, real value. There can be no fact of the matter about whether the temperature is, say, 29.00000001°C or 29.00000002°C. Consider $T_p$, the temperature at a point $P$. The best physical interpretation of $T_p$ is the average kinetic energy of the gas molecules in an imaginary sphere centered at $P$. As Smith points out, although the size of the sphere over which the average is taken is finite, it need not have any particular size. Whatever radius happens to be chosen is arbitrary; it could take an infinite number of values within some bounded range. Moreover, one must simply choose how to treat molecules crossing the border of the sphere: as if they were captured within the boundary, as if they were completely outside, or split the difference in some way. This arbitrariness supports the argument that there is no fact of the matter about the precise value of $T_p$.

A similar story can be told with respect to virtually any state variable used in a chaotic model if quantum mechanical fluctuations are considered. As others have claimed, there is no fact of the matter about the precise position, momentum, energy, etc., of any system when one considers quantities on the scale of Planck’s constant (see for example Paul Teller [1979], p. 352). Nonetheless, in order to accommodate fractal structure in the phase portrait, chaotic models specify real-numbered values for each of these physical magnitudes. In so doing, the models misrepresent the facts. As Smith puts it,

[In the typical case, the very thing that makes a dynamical model a chaotic one (the unlimited intricacy in the behaviour of possible trajectories) can not genuinely correspond to something in the time evolutions of the modelled physical processes—since they can not exhibit sufficiently intricate patterns at the coarse-grained macroscopic level ([1998], p. 41).]

How then can one justify this nonphysical structure? Smith considers several solutions to the problem before proposing his own. Their defects show that interpreting chaotic mathematical models realistically is uniquely difficult.

One might start with the fact that using false properties is a common practice in working science. Perhaps those properties associated with strange attractors do not need a new, creative justification. Fractals in dynamics might be on a par with, say, charge densities in electrostatics. One could then tailor the justification used for idealizations in general to fit the special cases found in chaos.

Smith agrees that garden-variety idealizations pave the way for simplified mathematics and this simplification is the key to justifying their use. However, he points out that frictionless planes and massless pendulum rods simplify by ignoring a property that exists in the real-world system.
Mathematical tractability is gained by abstracting away something that is actual. In contrast, the problem with fractal attractors is that they have too much detail. Chaotic mathematical models appear to posit ‘unlimited and necessarily non-empirical fine structure’ ([1998], p. 43), adding properties rather than ignoring what is there. So the question remains, compared to the lean underlying physics, how can an overly complex fractal model be a good choice for the modeler?

Another approach is to ignore the idealizations and appeal directly to the mathematics. After all, the first great promise of chaos was its potential for capturing unpredictable behavior—like the onset of turbulence—with a small set of equations (such as [1], the well-known Lorenz equations). Smith agrees that tractable dynamical equations can ‘specify infinitely intricate solutions as easily as they can specify e.g. nice elliptical solutions’ ([1998], p. 43). Perhaps the geometrical intricacy of a chaotic phase portrait is the price one pays for analytical simplicity in the mathematics.

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= -bz + xy
\end{align*}
\]

(1)

In Smith’s view, this direct appeal to simplicity in the equations is precluded by the qualitative approach to ODEs (see Arnold [1983] and Devaney [1989]). Equations like (1), he argues, merely specify a bundle of trajectories in phase space. It is the phase portrait, not the equations, that represents the evolution of a real system. Modelers do not compare equations to dynamical systems. They instead check the behavior of the system against the properties of the phase portrait. Smith takes this to imply that the semantic content of the models is primarily geometrical, not analytical. If mathematical models are (at least approximately) true, then the geometrical properties of the phase portrait are the truth-bearers, not the equations. Hence, the desire for more tractable equations cannot justify the nonphysical structure in a strange attractor. Something more is needed.

At this point, the stock answers have been exhausted and Smith proposes a unique strategy. First, he rejects the inclination to think of fractals in terms of ‘monstrous complexity’. Consider the Sierpinski Triangle (Figure 1). Although this fractal contains self-similar structure at all scales, its complexity becomes more manageable if one realizes how the figure is constructed.

Consider the first four iterations of a simplified Sierpinski triangle (Figure 2). The process begins by removing an inverted triangular section from the middle. Three white triangles remain, each with a side half the length of the original. The process is repeated for each of these three triangles, and so on to infinity. The end figure is a complex fractal—the product of countably infinite iterations. However, the operation used to produce it was quite simple.
Here, then, is where Smith finds the needed simplicity in chaotic models. Instead of focusing on the attractor itself in all its complexity, one should see its structure as the result of simple stretching and folding operations on sheets of phase portrait trajectories. The nonlinear ODEs associated with a given chaotic phase portrait specify simple stretch-and-fold operations which in turn leave an invariant fractal in their wake. Strictly speaking, this stretching-and-folding of sheets of trajectories is what the modeler is trying to capture:

[S]uppose we concentrate on modelling the way that the dynamics of some phenomenon basically stretches apart and folds back trajectories (trajectories, remember, of state-representing points in some suitable phase space). We will no doubt precisify and simplify, but at this stage in an entirely standard way [i.e., ODEs and real numbers will be used]. However, despite the simplicity here, a fractal may yet drop out as the invariant attractor ([1998], p. 50).

It is not that infinite structure is imposed in the modeling process, Smith concludes; it ‘comes for free’ via the iterations of stretching-and-folding trajectories (henceforth, SFT). Still, in order to apply the model, one must ‘defictionalize’ the precise state values since, again, systems in reality cannot support the infinite precision found in real numbers. (On the other hand, more ‘robust’ properties that do not depend on precise initial conditions—
e.g., period doubling as parameter values change—can be interpreted realistically as they stand.)

In other words, Smith believes that ODEs and real-valued state variables are useful for encoding SFTs in a mathematical model. They are the best tools available for specifying the behavior of orbits in the phase portrait, which are the subjects of the model. Fractal attractors are—borrowing a term from Mary Hesse—‘negative analogies’ between the model and the system being modeled. Once the experimentalist applies the model and defictionalizes such properties, they fall away as uninterpreted geometrical debris.

I agree that the notion of ‘imposing’ an unlimited amount of structure on a comparatively lean physical system is not the best way to account for the presence of fractal attractors. Recall, however, that Smith rejects several approaches to the problem because of their inability to account for this imposition of complex structure. I argue in section 4 that if one rejects the notion of imposing structure from the start, more natural ways of solving the problem of infinite structure present themselves. Before examining how this might go, let us consider three shortcomings of Smith’s proposed solution.

3 ODEs, eliminating fictions, and the exploitation of phase space

There are three reasons to be suspicious of Smith’s account, one having to do with the use of ODEs in contemporary dynamics, and the other two with its implications.
3.1 Discovery in qualitative dynamics

Smith sees SFTs as the central feature of chaos. This is the fundamental property he takes the modeler to be capturing; strange attractors are merely a by-product. The importance of ODEs (such as [1]) lies strictly in their ability to reproduce SFTs in their associated phase portraits.

One weakness in this view is that it misidentifies the subject of research and discovery in nonlinear dynamics. Consider for example the well-known case of Edward Lorenz in the 1960s (Lorenz [1993], p. 130). In order to prove the limitations of statistical weather forecasting using linear models, Lorenz used a set of nonlinear ODEs with twelve variables to describe the ‘weather’ in an idealized dynamical system representing the atmosphere. His goal was to produce simulated weather data that linear models could not replicate. Lorenz discovered that his equations displayed sensitive dependence on initial conditions (SDIC).\footnote{Roughly, if a system displays SDIC, then its future state changes dramatically given an arbitrarily small change in initial conditions. In terms of the system’s phase portrait, SDIC means that nearby trajectories diverge exponentially over time.} He eventually simplified the equations to get [1] and then plotted the numerical values for the state variables. The figure produced was a rudimentary form of the strange attractor now known as the Lorenz Mask. The qualitative details of the attractor and its phase portrait, including SFTs, were uncovered in subsequent research papers (see Kaplan and Yorke [1979] and Williams [1979]).

The point is that SFTs were latecomers in the chain of discovery. As such, it is difficult to see it as the subject of chaotic modeling, as the thing that researchers are seeking to capture with equations. Dynamicists seldom start with SFTs and then look for equations that can reproduce it. Instead, one typically begins with a physical model of a real-world system that is amenable to description by ODEs. The discovery of SFTs themselves, usually associated with Stephen Smale’s ‘horseshoe’, bears this out. Smale was conducting research on van der Pol’s equation at the time (Diacu and Holmes [1996], pp. 55–65). This equation was well known long before anyone came to understand the complex structure of its phase portrait.

Nonlinear systems theorists typically do not try to model SFTs, nor are ODEs used merely because they are the best vehicle available to capture geometrical structures in phase space. It seems, therefore, that SFTs do not belong at center stage either historically or as a rational reconstruction of dynamical modeling.

3.2 ‘Defictionalizing’ strange attractors

The next question is whether strange attractors can be explained away in the prescribed manner. Smith’s treatment bears a family resemblance to that of
Prigogine, who has long argued that state trajectories are idealizations that the experimentalist must discard. But Prigogine is never mentioned. Smith focuses on fractal attractors, not the foundations of phase space and trajectories.

Can one take an anti-realist approach toward strange attractors? Yes, but at a price. If strange attractors cannot be interpreted realistically—if chaotic aperiodicity is always a negative analogy to the dynamical system—then actual dynamical systems must be governed by something else. There are three remaining categories of attractors in dissipative models, viz. point, periodic, and quasiperiodic attractors. Let us consider these in order. Point attractors describe systems that eventually come to rest, like a marble in a cup. Next, periodic attractors (or limit cycles) are found in the phase portraits of damped, driven, periodic systems such as a clock pendulum. Orbits in the basin of attraction of a periodic attractor form closed curves as \( t \to \infty \). Finally, trajectories on a quasiperiodic attractor densely cover the surface of \( n \)-tori in phase space without closing. This aperiodic attractor is the most complex of the three, but it is not a fractal (Tabor [1989], pp. 195–202).

What each of these attractors lacks is important. In none of the three cases is there exponential separation of nearby trajectories and therefore no SDIC. Many would argue—and rightly so, I believe—that SDIC is the heart of chaos theory. If neither strange attractors nor SDIC are real phenomena, then chaos is essentially taken out of the explanatory resources of applied dynamics. This would seem to be a steep price indeed. Nonetheless, in his earlier paper Smith was willing to pay it:

[I]f locations along dimensions of the phase space are supposed to represent physical quantities like (say) temperature and fluid velocity, then it makes no physical sense to suppose that there is really infinitely sensitive dependence (or a corresponding real fractal structure) in nature ([1991], p. 266).

The difficulty with this view is that strange attractors and SDIC have become indispensable. Researchers have known since the early 1980s that long-term periodicity and quasiperiodicity cannot account for all of the phenomena it was once thought. Experimentalists now routinely distinguish chaotic from non-chaotic time series data using a variety of diagnostic tools. For example, power spectrum analysis can often rule out periodic and quasiperiodic attractors. Phase space reconstruction and the calculation of Lyapunov exponents can confirm the presence of a strange attractor (Koperski [1998]). In short, researchers can no longer massage chaotic time-series data into the categories of linear dynamics. If strange attractors and SDIC are not real, then an entirely new account of dissipative chaos is needed.

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3 Most recently in Prigogine ([1996], p. 105).
3.3 The special exploitation argument

Recall the close conceptual link between the infinite structure problem and the use of real numbers mentioned earlier (section 2). Smith argues that the time evolutions of actual state variables ‘cannot really exemplify infinitely intricate trajectories wrapping round a fractal attractor’ because the properties associated with those variables (e.g., pressure) ‘cannot have indefinitely precise real number values’. So without real numbers, there can be no fractal attractor.

In one sense, this is clearly true. Phase spaces that contain strange attractors are typically (but not always) Euclidean, $\mathbb{R}^n$ manifolds. Remove the $\mathbb{R}^n$ structure, and you remove the canvas on which strange attractors are painted. But then again, this also takes away the canvas on which all attractors are painted, chaotic or not. Point attractors and limit cycles are just as much creatures of phase space as strange attractors. If there is something wrong with real-valued state variables, the problem lies in the qualitative approach to contemporary dynamics, not just chaos.

Smith himself realizes this and presses on to narrow his target. What strange attractors have that others lack is fine structure all-the-way-down (i.e., on all scales in phase space), a consequence of their fractal geometry. More precisely, the attractor is topologically transitive. A map $f: \mathbb{R}^n \to \mathbb{R}^n$ is topologically transitive if for any pair of open sets $A, B \subseteq \mathbb{R}^n$ there is a $k$ such that $f^{(k)}(A) \cap B \neq \emptyset$ (or, equivalently, for any two points in a dynamical system, there is an orbit that comes arbitrarily close to both (Devaney [1989], p. 49, [1992], p. 117). $k$ represents forward iterations of the map. An attracting set of phase space points is topologically transitive if there is a trajectory in the basin of attraction that visits every region in the limit set, no matter how small the region. Since strange attractors are topologically transitive, a trajectory on the strange attractor explores every region of the attracting set as $t \to \infty$.

This, then, is a slightly more precise interpretation of fine structure at all scales. Chaotic evolutions exploit phase space by way of topological transitivity. Unfortunately, non-chaotic evolutions with the same property seem to have been overlooked. Quasiperiodic attractors are also topologically transitive. (This follows from the fact that trajectories on such attractors are dense on the torus (Devaney [1992], p. 117).) It seems, therefore, that any argument directed against the physical impossibility of chaotic fine structure must apply equally to quasiperiodic fine structure.

In slightly different terms, what I have tried to show in subsection 3.3 is this. Smith wants to prove that strange attractors cannot govern the behavior of real-world systems. One way to do this is to attack its fractal cross-sections as unrecognized fictions: nature cannot honor this fine structure all-the-way-
down. Specifically, one could argue that there are no physical grounds for a phase space fractal in which there is a trajectory cutting through every small neighborhood on the attractor. However, once we replace labels like ‘fine structure’ and ‘intricacy’ with their more precise counterpart, topological transitivity, we find that quasiperiodic evolutions have it as well. In this regard at least, chaos offers nothing new. Strange attractors do not exploit the $\mathbb{R}^n$ structure of phase space to any greater degree than quasiperiodic attractors.

Once topological transitivity and quasiperiodicity are in the picture, it appears that chaotic attractors have drawn undue attention. The Cantor set-like properties of a strange attractor and the densely wrapped tori of a quasiperiodic attractor both have structure at all scales. Lacking a more focused argument to the contrary, chaotic attractors appear to be no more problematic than the quasiperiodic variety.\(^6\)

So then, what has gone wrong? Smith calls for a ‘new perspective’ after having rejected the standard views on the use of idealizations. I argue that those approaches were tossed aside too quickly. The best way to account for fractal structure might not be a stock solution, but neither does it require something completely novel.

4 Wilson, Truesdell, and continuum mechanics

Given that modern chaos theory has only been around for thirty years or so, it is surprising to find similar issues in the more mature realm of continuum mechanics. Still, this is where we find the clearest cases of nonphysical surplus structure in idealized dynamical systems. Seeing how the problem is resolved in continuum models will suggest a strategy for chaotic models. This approach is motivated by the work of philosopher of science Mark Wilson, and Clifford Truesdell, the dean of twentieth-century rational mechanics.

4.1 Wilson

Wilson seems to have something akin to Smith’s defictionalizing of physically impossible states in mind when he writes, ‘[I]f left solely to its own devices, the heat equation accepts mathematical solutions that we discard as “unphysical” precisely because these solutions deposit implausible boundary conditions upon their frontier’ ([1991], pp. 567–8). Although the heat equation is (at least approximately) true, insight from the actual physics involved is required in order to apply it to dynamical systems. The equation

\(^6\) Elsewhere, Smith suggests that all models in classical macrophysics idealize in much the same way ([1998, 71]). All such mathematical models, therefore, are at best approximately true. However, if this is intended to be the ultimate answer to the problem, then it is not clear why such a novel interpretation of fractal fine structure was needed in Chapter 2.
permits solutions that, although perfectly good from a mathematical point of view, must be rejected because the underlying physics disallows them.

Let us consider a simple example that Wilson uses to make his point. In introductory continuum mechanics, the end-points of a violin string are considered fixed. This idealization simplifies the equations involved. We know, however, that real violin strings cannot have fixed endpoints. If that were the case, no energy would be transferred to the body of the violin and therefore no sound would be produced. Strictly speaking, the equations ‘predict’ that the violin will not produce any music. However, this artefact of the fixed-point idealization is easily spotted and dismissed.

A brief word about the terms used here is in order. An *idealization* is a simplifying assumption made before one constructs a dynamical equation or mathematical model. For instance, when engineers decide to ignore the friction in a newly designed component, they do so because friction is negligible for the application at hand. The engineer has idealized away the damping force and so the associated mathematics is simplified—likewise in considering the planets to be point masses, transmission lines to be noiseless, and all the rest. *Artefacts*, in contrast, are the false properties or relations that can result from idealizations. An artefact is not an abstraction built into the model; it is a (possible) consequence of simplifying assumptions. Artefacts are often benign. No one is confused when the Census Bureau refers to the 2.14 children in the average American home. At times, however, it is difficult to separate the artefacts from genuine physical properties, or in Hesse’s terms, to figure out whether the neutral analogies are in fact positive or negative. This is especially so in the mathematical sciences where uncovering the physical significance of certain terms is often nontrivial.

One way of interpreting Smith’s conclusion is this: the fractal structure of a strange attractor is a nonphysical, artefactual result of using real-valued state variables and a Euclidean phase space. (This is very much the line of Smith’s early thought on the problem [1991].) In this light, his concerns take on a familiar form. Most mathematical models contain idealizations. Some of these produce artefacts. The artefact might require a change in the model or it might simply be ignored. In the case of the violin, we choose the latter knowing that energy must be transferred from the strings to the body of the instrument. Similarly, the argument goes, one should not interpret the structure of a strange attractor realistically. Doing so can only lead to conflicts with known facts about the underlying physics.

The analogy between violin and attractor breaks down, however. Although the fixed end-point boundary condition is false, it allows for solutions that get the dominant behavior of the string correct. The simplified equation closely approximates the dynamics of real violin strings. For a more accurate model, a more realistic version of the fixed-string boundary is needed. (Specifically,
the boundary conditions will be moved into the body of the violin in order to capture the vibration of the end-points.) Of course, the move toward greater realism entails greater mathematical complexity. The new, less idealized equation will also be less tractable.

In contrast, if Smith is correct, then strange attractors are already too complex. The path to greater realism lies in taking these overly precise models and making them less so. Hence, it does not appear that the problem of infinite structure fits under Wilson’s more general approach toward dealing with artefacts.

Wilson’s contribution to the subject does not end here. More importantly, his account suggests a straightforward approach to idealized surplus structure. First, one must consider a slightly different set of questions. Why do dynamicists use real-numbered state variables? Why consider the mass in the violin string a continuum rather than as molecular? Why impose artificial boundary conditions when no strict boundary exists in nature? The answer to each of these questions is clear: because we want to be able to use differential equations. The power of the mathematics drives all branches of dynamics.\(^7\) Truesdell and Muncaster have a similar view:

However discrete may be nature itself, the mathematics of a very numerous discrete system remains even today beyond anyone's capacity. To analyze the large, we replace it by the infinite, because the properties of the infinite are simpler and easier to manage. The mathematics of large systems is the infinitesimal calculus, the analysis of functions which are defined on infinite sets and whose values range over infinite sets. We need to differentiate and integrate functions. Otherwise we are hamstrung if we wish to deal effectively, precisely, with more than a few dozen objects able to interact with each other. Thus, somehow, we must introduce the continuum (Truesdell and Muncaster [1980], pp. xvi–xvii).

Although this passage is about continuum mechanics and partial differential equations, the same line of thought applies to nonlinear dynamics and ODEs. In this case, the move from the discrete to the infinite occurs in ignoring tiny, discontinuous jumps in the state variables. Given the microphysics involved, precise values for classical quantities might change discontinuously. Nevertheless, for well-behaved equations to describe their evolution, the state variables must take on real values and their change of state must be (piecewise) smooth.

All this suggests that focusing on strange attractors, fractal structure, trajectories, and the rest is misguided. These geometrical entities are useful only insofar as they provide qualitative insight into the governing equations.

\(^7\) In fact, Wilson ([1991]) describes several ways mathematicians deal with idealized initial and boundary conditions. The direct appeal to mathematical convenience is, as Wilson says, ‘sometimes warranted’, but in other cases a more involved story must be told.
The properties of the phase portrait—chaotic and otherwise—are completely determined by the equations associated with that space. If one uses an unwarranted idealization in order to set up the ODEs, then that is where the argument needs to be directed. State variables in chaotic models might take on physically impossible values. Nonetheless, the root of the problem has nothing to do with strange attractors. If there is a fundamental mismatch between the interpretation of a phase portrait and the real-world system, the blame must lie in the differential equations whose solutions are represented in that space. Smith, I believe, is taking aim at the wrong target. His concerns point not to chaos proper, but rather to ODEs and their standard idealizations.

Smith clearly does not intend a broad skeptical attack on differential equations. Can a more focused, cogent version of the argument be made? In my view, no. It is extremely difficult to argue for a narrow anti-realism about strange attractors that does not encompass a great deal in modern dynamics.\(^8\)

Let us now consider a second, related perspective on nonphysical surplus structure.

### 4.2 Truesdell

Recall once more the way Smith proposed the problem of infinite structure. Unlike textbook physical models such as the ideal pendulum, chaotic models appear to add an infinite amount of surplus structure that does not exist in reality: `there is too much detail rather than too little; they are opulent rather than impoverished` ([1998], p. 42). There is therefore a disanalogy between the fractal structure in a strange attractor and run-of-the-mill idealizations in dynamics.

Chaotic models are not the first to seemingly impose surplus structure on reality. Consider this example from Tim Poston and Ian Stewart:

[Fluid mechanics] supposes that the velocity of the fluid at each point \( x \) can be given by a vector \( \mathbf{v}(x) \), usually varying smoothly with \( x \). Strictly, this is nonsense. On a small enough scale we would see molecules bouncing off each other with highly varying velocities, and empty spaces between them where no velocity could be assigned (Poston and Stewart [1978], p. 217).

In other words, the models used in fluid mechanics seem to assert a fact-of-the-matter regarding fluid velocity where there is none. For such models to be correct, real fluids would have to form a continuum at all scales. Of course,

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\(^8\) Smith is, at times, on the verge of just such a conclusion. Officially, all macroscopic dynamical models are approximately true, rather than true *simpliciter* ([1998], p. 71). It is not clear, however, how this squares with the uniquely chaotic problem of infinite structure.
the molecular nature of matter is inconsistent with such facts. How, then, is the imposition of this surplus content justified?

The short answer is: it isn’t. One need not take the models as adding surplus structure to the relatively lean physics of molecular fluids. Instead, as Truesdell vigorously argues, the truth lies in the opposite direction. Rather than imposing a false microstructure on the world, continuum models like those in fluid mechanics ignore the small-scale facts:

Widespread is the misconception that those who formulate continuum theories believe matter ‘really is’ continuous, denying the existence of molecules. That is not so. Continuum physics presumes nothing regarding the structure of matter. It confines itself to relations among gross phenomena, neglecting the structure of the material on smaller scales (Truesdell [1984], p. 54).

Strictly speaking, there can be no mismatch between facts in the macroscopic model and facts in the microscopic world. For there to be a mismatch, continuum models would have to imply something about the structure of matter. However, nothing is presumed and nothing is implied. Engineers using continuum mechanics are not concerned with the makeup of the material in their models. If pressed, the engineer will surely concur that the air flowing over a wing is composed mostly of nitrogen molecules; nonetheless, the equations governing that airflow ignore this fine structure. For the application at hand, the microstructure simply does not matter. The microscopic components underdetermine the properties of the macroscopic material, but the utility of continuum models in no way hinges on our ability to resolve this underdetermination. At bottom there might just as well be Newtonian corpuscles, Bosovichian point masses, or Leibnizian monads, so long as they behave in large numbers like a continuous fluid.

This, then, is why no justification is needed for the surplus material in continuum models. Far from imposing an infinite amount of structure where there is none, the correct view is that the small-scale structure is ignored.

Returning now to chaos, recall that the question Smith is out to answer is how models positing ‘unlimited and necessarily non-empirical fine structure’ can ever be a good choice for the modeler. Although this seems reasonable, I believe that the best response is to refuse the assumptions behind this loaded question. If at the onset we resist the urge to see the problem as positing or projecting nonphysical structure, then a solution that Smith initially rejects turns out to be close to the mark.

Consider the temperature example once more (section 2). Smith argues that the fractal structure of a strange attractor forces $T_p$ into taking nonphysical values. If there is no fact of the matter regarding $T_p$ insofar as it maps to precise, real numbers, then there is an obvious mismatch between the world and the model. How then do we account for this? Following Truesdell, one
ought not to infer anything as to what \( T_p \) qua state variable corresponds to at the molecular level. Again, atoms, point masses, or monads, it does not matter, so long as macroscopically detectable changes in \( T_p \) are continuous rather than discrete. Crudely put, if \( T_p \) flows rather than jumps from one value to another, then real numbers are appropriate. Points and trajectories in an \( \mathbb{R}^n \) phase space follow close behind. The fundamental decision for the modeler is whether changes in \( T_p \) at a certain scale can be described in a phase space. It does not matter that at some level nature fails to sanction the use of real numbers. The question is whether \( T_p \) behaves that way at the scale the modeler is interested in. Of course, there is nothing special about \( T_p \). The same goes for any physical trait.

Furthermore, the possible values of the state variables—represented by the phase space points—are not influenced by whether the model is chaotic or not. No matter how the system evolves—periodically, quasiperiodically, or chaotically—the range of physically possible states remains the same. The dynamics cannot add or impose anything. Therefore, fractal attractors do not force the model into nonphysical values. Whether a given state is physically possible or not is fixed within the phase space without regard to the system’s evolution. Trajectories on a strange attractor can only move through state points that are ‘already there’. If phase space models take on impossible values, the blame must lie with the equations that govern that space and the idealizations used to set it up.

5 Conclusion

We should note once more that Smith’s own approach does not imply that chaotic models impose nonphysical structure. However, he uses this notion in his critique of several common strategies for dealing with idealizations and artefacts. Only then does he give his novel account of fractal attractors as an epiphenomenon of the modeling process. In response, I have argued for two things.

First, although Smith rejects global skepticism toward dynamical models, it is difficult to say what is wrong with chaotic models without having that criticism spill over into the rest of dynamics. In particular, since quasiperiodic attractors share many of the characteristics of strange attractors, if the latter variety is problematic, so are their non-chaotic cousins. Second, Smith’s solution to these problems is more creative than need be. If at the onset we resist seeing fractal structure as adding a complex, nonphysical property to phase space models, then the standard accounts of idealizations that Smith rejects are viable. As we have seen, the argument that fractals are not like frictionless planes—since only the latter suppress an existing property—is
misleading. Truesdell in particular helps identify where and why macroscopic models ignore small-scale phenomena.

In the end, there is not much new under the sun presented by chaos theory. Smith agrees with this perspective throughout Explaining Chaos, except when it comes to strange attractors. I suggest the same attitude be taken there as well.

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Department of Philosophy
Saginaw Valley State University
7400 Bay Road
University Center, MI 48710, USA
koperski@svsu.edu

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