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Russell's Paradox

(p. 45)

Statement of the paradox. Some sets are not members of themselves. For examples: The set of cats is not a cat. The set of pencils is not a pencil. Other sets are members of themselves. For examples: The complement of the set of cats (i.e. all non-cats) is not a cat, so is a member of itself. The set of all sets described on this page is a set described on this page, so is a member of itself. A library catalog that lists all the holdings of the library would list itself since it is itself a holding of the library; hence it is a member of itself.

Consider now the collection of all sets that are not members of themselves. This collection is either a member of itself or it is not a member of itself (since any set is well-defined). If the set is a member of itself, then it would belong to the collection of all sets that are not members of themselves, hence would not be a member of itself. If the set is not a member of itself, then it would belong to the collection of all sets that are not members of themselves, hence would be a member of itself. In either case there is a contradiction.

Related paradoxes. Russell's paradox is akin to two other logical paradoxes: The Cretan Epimenides said that all Cretans are liars. A sheet of paper has printed on one side 'The statement on the other side of this paper is false', and on the other side has printed 'The statement on the other side of this paper is true'.

Origins and early history. The paradox was discovered in May, 1901 by the celebrated mathematician and logician Bertrand Russell while he was preparing the first draft of *Principia Mathematica*. He was investigating Georg Cantor's proof that there is no greatest number. Earlier Cesare Burali-Forti discovered a similar paradox when he noticed that the set of ordinal numbers is well-ordered, so it too must have an ordinal number.

Russell wrote to Gottlob Frege about his paradox in June, 1902. This news disturbed Frege, since the import of the paradox was that the axioms that Frege was using to formalize his logical theory were inconsistent. Unfortunately Frege's book was in press, but he had to abandon many of his views as a consequence of the paradox.

Implications. By logic, all statements follow from a contradiction; ergo all mathematical proofs are suspect since they rest on logic and set theory. Another dire result was that Cantor's axiom of unrestricted comprehension had to be modified or abandoned. (This axiom states that any predicate expression P(x) containing a free variable determines a set {x: P(x)} where we have used Giuseppe Peano's set-builder notation.)

Russell's solution. The theory of types was Russell's solution to the problem. Here is one explanation: Players form a football team; the football teams form a league; the leagues form a conference; the conferences form an association. Note that a player cannot be a team or a league or a conference or an association, since he is the *wrong* type of thing for these memberships.

Similarly, individuals (type C) are members of sets (type 1); sets are members of classes (type 2); and so on. Thus a hierarchy results, and only the adjacent levels mix on the level of $\in$-membership; such membership does not exist within a level.

Russell's solution was criticized for being ad hoc. Other solutions proposed were David Hilbert's formalism, Luitzen Brouwer's intuitionism, and Ernst Zermelo's axiomatization of set theory.

Sources. The website offered these three references:
The Stanford Encyclopedia of Philosophy.
The Philosophers' Magazine, article by Francis Moorcroft.
Cut the Knot! website.