Counting

Below the title bar for each case is a list of models that exemplify that case. The exception is with inclusion-exclusion variant; there significant applications are presented.

Method: Tree Diagram

Especially useful in counting cases where the diagram is not symmetric.

Method: Addition Rule (Fundamental Principle of Counting)

1. If a set is formed by sequence of tasks, of which the first can be performed n_1 different ways, the second in n_2 different ways, . . . , the last in n_k different ways, then the set contains $n_1 n_2 \ldots n_k$ different elements.
2. $|A_1 \times A_2 \times \ldots \times A_k| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_k|$

Method: Sum Rule

1. If a set is formed by one of k mutually exclusive tasks, and the first task can be performed in n_1 different ways, the second in n_2 different ways, . . . , the last in n_k different ways, then the set contains $n_1+n_2+ \ldots +n_k$ different elements.
2. If $A_1, A_2, \ldots, A_k$ are mutually exclusive sets, then $|A_1 \cup A_2 \cup \ldots \cup A_k| = |A_1| + |A_2| + \ldots + |A_k|$

Method: Pigeonhole Principle

1. (Basic) If k+1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.
2. (Generalized) If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Permutation (distinguishable elements, no repetition)	$_nP_r = n \cdot (n-1) \cdot \ldots \cdot (n-r+1) = \frac{n!}{(n-r)!}$ $0 \le r \le n$

1. From set S containing n distinguishable elements, select r elements without repetition, then arrange the r elements in a row.
2. Draw without replacement r balls from an urn containing n distinguishable balls, then arrange them in a row.
3. One-to-one functions from domain containing r elements to codomain containing n elements.

<table>
<tr><td>Permutation
(distinguishable elements, repetition)</td><td>n^r</td></tr>
<tr><td colspan="2">1. From set S containing n distinguishable elements, select r elements (permitting repetition), then arrange the tokens of the r elements in a row.
2. Draw with replacement r balls from an urn containing n distinguishable balls, then arrange their tokens in a row.
3. Functions from domain containing r elements to codomain containing n elements.</td></tr>
<tr><td>Permutation
(elements divided into k types)</td><td>$\left(\begin{matrix} n \\ n_1\ n_2 \ldots n_k \end{matrix}\right) = \left(\dfrac{n!}{n_1!n_2!\ldots n_k!}\right)$</td></tr>
<tr><td colspan="2">1. Arrange into a row the elements of set S which contains n elements of which n_1 are of type 1, n_2 are of type 2, . . . , n_k are of type k. Elements of the same type are indistinguishable.
2. Place n indistinguishable elements into k distinguishable bins so that n_1 elements are in bin 1, n_2 elements are in bin 2, . . . , n_k are in bin k.</td></tr>
<tr><td>Combination
(elements distinct, no repetition)</td><td>$_nC_r = C(n,r) = \dfrac{n!}{r!(n-r)!}$
$0 \le r \le n$</td></tr>
<tr><td colspan="2">1. From set S containing n distinguishable elements select r.
2. Draw without replacement r balls from an urn containing n distinguishable elements.</td></tr>
<tr><td>Combination
(elements divided into types, repetition)</td><td>Stars & bars C(n+r-1,r)</td></tr>
<tr><td colspan="2">1. From set S containing n types of elements, r are drawn. Within each type, elements are indistinguishable.
2. Number of non-negative integer solutions of $x_1+x_2+ \ldots +x_n=r$.
3. Place indistinguishable balls into distinguishable bins.</td></tr>
<tr><td>Inclusion-Exclusion
(elements distinguishable)</td><td>$|A \cup B \cup C| = |A| + |B| + |C|$
$-(|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|$
Extends to more than 3 sets.</td></tr>
<tr><td colspan="2">Count number of elements in union of non-disjoint sets.</td></tr>
<tr><td>Inclusion-Exclusion Variant</td><td>$N(P_1'P_2' \ldots P_n') = N - |A_1 \cup A_2 \cup \ldots \cup A_n|$</td></tr>
<tr><td colspan="2">Applications: 1. Integer equations with constraints.
2. Sieve of Eratosthenes.
3. Onto functions.
4. Derangements.</td></tr>
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