MATH 300 (2028) Test 3 2002 May 2

Name ANSWER KEY ______points of 159 ______%

Write answers and show all work on these sheets. Since partial credit will be given, show sufficient detail. The number of points for each question is shown in parentheses after the number of the question.

1. (8) For each of the following relations, determine if it is (1) reflexive, (2) symmetric, (3) transitive, (4) antisymmetric.

a. Set = $\mathbb{Z}$; aRb means $a^2=b^2$.

(1) $a^2=a^2 (\forall a)$ $\therefore$ reflexive.

(2) $a^2=b^2 \Rightarrow b^2=a^2 (\forall a,b)$ $\therefore$ symmetric.

(3) $a^2=b^2 \wedge b^2=c^2 \Rightarrow a^2=c^2 (\forall a,b,c)$ $\therefore$ transitive.

(4) $(-1)^2=1^2$ and $1^2=(-1)^2$ yet $-1\neq 1$. $\therefore$ not antisymmetric.

b. Set = Humanity; aRb means *a is at least as tall as b*.

(1) A person is at least as tall as himself. $\therefore$ reflexive.

(2) a may be taller than b, so aRb and b$\not R$a. $\therefore$ not symmetric.

(3) a is at least as tall as b $\wedge$ b is at least as tall as c $\Rightarrow$ a is at least as tall as c $\therefore$ transitive.

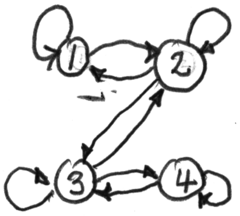
(4) Two different people can be the same height. $\therefore$ not antisymmetric.

2. (15) For set {1,2,3,4}, aRb means $|a-b|\le 1$. Find the

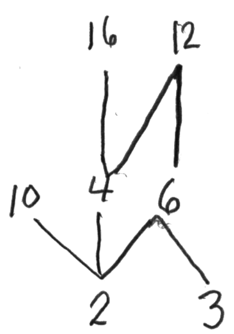
a. matrix $M_R = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$

b. matrix for $\mathbf{R}^2 = M_R^{[2]} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}}$

c. graph of **R**:



3. (7) Draw the Hasse diagram for the set ={2,3,4,6,10,12,16}, aRb means a|b.



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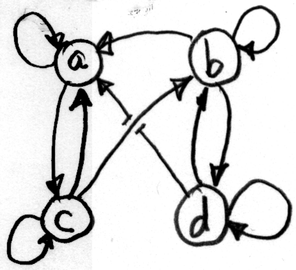
4. (22) For the relation matrix $M_R = \begin{array}{c|c} & \begin{matrix} a & b & c & d \end{matrix} \\ \hline & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{matrix} a \\ b \\ c \\ d \end{matrix} \end{array}$

a. (5) Draw the directed graph.



b. (4) Determine if **R** is (1) reflexive, (2) symmetric, (3) transitive, (4) anti-symmetric.
(1) ALL 1's on principal diagonal. ∴ reflexive.
(2) Matrix is not symmetric, e.g., $a_{12} = 0 \neq 1 = a_{21}$. ∴ not symmetric.
(3) bRa and aRc, yet $b \not R c$. ∴ not transitive.
(4) aRc and cRa, yet $a \neq c$. ∴ not antisymmetric.

c. (3) Give the matrix for the reflexive closure of **R**.
d. (3) Give the matrix for the symmetric closure of **R**.
e. (7) Give the matrix for the transitive closure of **R**.

c. M_R, since R is already reflexive

d. $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \vee \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}}$

e. $M_T = M_R + M_R^{[2]} + M_R^{[3]} + M_R^{[4]} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \vee \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \vee \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}}$

Notice that $M_R^{[4]}$ was not needed.

5. (5) Set = $\mathbb{N}$. aRb means that a ends with the same digit as b. Is this an equivalence relation?

(1) a ends with the same digit as itself. ∴ reflexive.
(2) If a ends with the same digit as b, then b ends with the same digit as a. ∴ ~~reflexive~~. Sym.
(3) If a ends with the same digit as b, and b ends with the same digit as c, then a ends with the same digit as c. ∴ transitive.
Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

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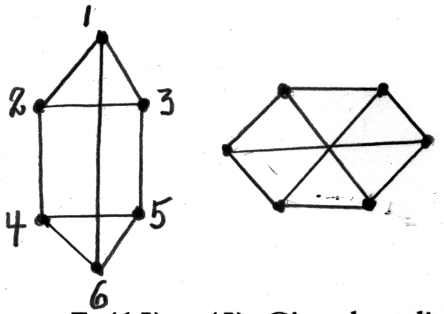
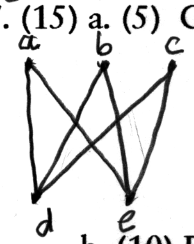
6. (8) Consider the two graphs below. If they are isomorphic, label the vertices of the right graph to show the isomorphism. If they are not isomorphic, prove it.



Not isomorphic, because left graph has 3-cycles, e.g., 123, whereas graph on right has no 3-cycles.

7. (15) a. (5) Give the adjacency matrix for $K_{3,2}$.



$$A = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array}$$

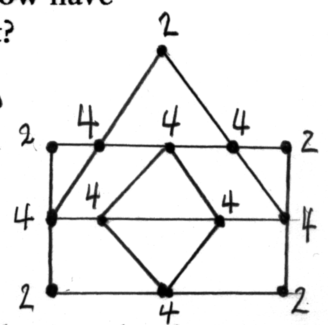
b. (10) Find the number of paths of length 2 between any two vertices of $K_{3,2}$.

$$A^2 = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right) = \boxed{\left(\begin{array}{ccc|cc} 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 \end{array}\right)}$$

8. (8) Does the graph below have

a. an Euler circuit?

Since each vertex is of even degree (as shown), it has an Euler circuit.



b. an Euler path?

Since each Euler circuit is an Euler path, yes.

9. (21) Suppose that you have 30 books (15 novels, 10 history books, 5 math books); assume that all books are different. In how many ways can you

a. (2) put the 30 books in a row on the shelf? $\boxed{30!}$

b. (3) take a bunch of 4 books to give to a friend? $\boxed{C(30,4)} = 27,405$

c. (4) take a bunch of 3 history books and 7 novels to give to a friend? $\boxed{C(10,3) \cdot C(15,7)}$ 120 · 6435 = 772,200

d. (5) put the 30 books on the shelf so that all the novels are on the left, the math books in the middle, and the history books to the right? $\boxed{(15!)(10!)(5!)}$

e. (7) put the 30 books on the shelf so that all books of the same kind are together?

$\boxed{(3!)(15!)(10!)(5!)}$

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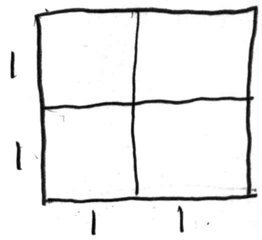
10. (15) Show that if 5 points are picked in the interior of a square whose sides are of length 2, then there are at least two of the points no further than $\sqrt{2}$ apart.



Divide the square as shown. By the pigeon hole principle, at least one square will contain 2 points. These points can be no further apart than $\sqrt{2}$, which is the length of the diagonal of a smaller square.

11. (10) a. Find the number of solutions to $x+y+z = 32$, where x,y,z are non-negative integers.

$C(32+3-1,2) = \boxed{C(34,2) = C(34,32)}$

(32 beans and 2 dividers)

b. Same as (a.) except that $x \geq 7$ and $y \geq 15$.

$C(10+3-1,2) = \boxed{C(12,2) = C(12,10)}$

Apportion 7+15 beans first, then 10 beans and two dividers.

12. (10) You have 50 of each of the following kinds of jelly beans: red, orange, green, yellow. The jelly beans of each color are identical.

a. In how many way can you put all the jelly beans in a row?

Permutation with 4 classes of indistinguishable elts. $\boxed{C\binom{200}{50,50,50,50} = \dfrac{200!}{50!\,50!\,50!\,50!}}$

b. How many handfuls of 12 are possible?

12-combination of 4 classes of indistinguishable elts. $\begin{cases} r=12 \\ n=4 \end{cases}$ $C(12+4-1,12) = \boxed{C(15,12)}$

13. (10) Solve the recurrence relation: $a_n=5a_{n-1}-4a_{n-2}$, $a_0=1$, $a_1=0$.

$a_n - 5a_{n-1} + 4a_{n-2} = 0$

Characteristic equation:

$r^2 - 5r + 4 = 0$

$(r-4)(r-1) = 0$

$r = 1, 4$

$a_n = \alpha_1 1^n + \alpha_2 4^n = \alpha_1 + \alpha_2 4^n$

$1 = a_0 = \alpha_1 + \alpha_2 \cdot 4^0 = \alpha_1 + \alpha_2$

$0 = a_1 = \alpha_1 + \alpha_2 \cdot 4$

$\Rightarrow$ $1 = -3\alpha_2 \Rightarrow \alpha_2 = -1/3$

$\alpha_1 = -4\alpha_2 = 4/3$

$\boxed{a_n = 4/3 - \frac{1}{3} \cdot 4^n}$

14. (5) An office manager has 4 employees and 9 reports to be done. How may ways can the reports be assigned fo that each employee has at least one report to do?

The assignment function f: report → employee must be onto.

Number of such functions is $4^9 - C(4,1)3^9 + C(4,2)2^9 - C(4,3)1^9$

$= \boxed{186,480}$

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