MATH 300 (2028) Test 2 2002 Mar 22

Name ANSWER KEY ______ points of ~~106~~ 96 ______%

Write answers and show all work on these sheets. Since partial credit will be given, show sufficient detail. The number of points for each question is shown in parentheses after the number of the question.

1. (10) a. Write $(110101)_2$ in base 10 (decimal) expansion.

$(110101)_2 = 1\times2^5+1\times2^4+0\times2^3+1\times2^2+0\times2^1+1\times2^0 = 32+16+4+1 = \boxed{53_{10}}$

b. Write $(21)_{10}$ in base 2 (binary) expansion.

$21_{10} = 16+4+1 = 1\times2^4+0\times2^3+1\times2^2+0\times2^1+1\times2^0 = \boxed{10101_2}$

2. (10) a. Use the Euclidean algorithm to find the gcd (44,52). $= \boxed{4}$

$52 = 44\cdot1+8 \Rightarrow 8 = 52+44(-1)$

$44 = 8\cdot5+4 \Rightarrow 4 = 44+8(-5)$

$8 = 4\cdot2$

$= 44+(-5)(52+44(-1))$

$= 44(1+5)+52(-5) = 44\cdot6+52(-5)$

b. Express this gcd in the form 44s+52t (as a linear combination).

$\boxed{4 = 44(6)+52(-5)}$

3. (10) a. Find an inverse of 2 modulo 17.

$2x \equiv 1 \pmod{17}$

$2\cdot9 = 18 \equiv 1 \pmod{17} \Rightarrow 2^{-1} \equiv 9 \bmod (17)$ (Trial & error)

b. Solve $2x \equiv 7 \pmod{17}$

$9\cdot2x \equiv 9\cdot7 \pmod{17}$

$1\cdot x \equiv 63 \pmod{17}$

$\boxed{x \equiv 12 \pmod{17}}$

4. (5) Find the non-negative integer less than 28 represented by (3,5) = (a mod 4, b mod 7)

Equivalent to 3 (mod 4) $\wedge < 28$: 3, 7, 11, 15, 19, 23

Equivalent to 5 (mod 7) $\wedge < 28$: 5, 12, 19, 26

$\boxed{19}$ since $19 \equiv 3 \pmod 4$ and $19 \equiv 5 \pmod 7$

5. (8) Which integers are divisible by 5 but leave a remainder of 1 when divided by 3?

$x \equiv 0 \pmod 5$: 0, 5, 10, 15

$x \equiv 1 \pmod 3$: 1, 4, 7, 10

$\boxed{x \equiv 10 \bmod 15}$

43

6. (10) Compute:

$$\begin{pmatrix}1&0&0&1\\0&1&0&1\\1&1&1&1\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&1\\1&1\\1&0\end{pmatrix}=\begin{pmatrix}1+0+0+1 & 0+0+0+0\\ 0+0+0+1 & 0+1+0+0\\ 1+0+1+1 & 0+1+1+0\end{pmatrix}=\boxed{\begin{pmatrix}2&0\\1&1\\3&2\end{pmatrix}}=\boxed{\begin{pmatrix}1&0\\1&1\\1&1\end{pmatrix}}$$

3×4 $\quad$ 4×2 $\quad$ $\therefore 3\times 2$

$$\begin{pmatrix}1&0&0&1\\0&1&0&1\\1&1&1&1\end{pmatrix}\odot\begin{pmatrix}1&0\\0&1\\1&1\\1&0\end{pmatrix}=\begin{pmatrix}(1\wedge1)\vee(0\wedge0)\vee(0\wedge1)\vee(1\wedge1) & (1\wedge0)\vee(0\wedge1)\vee(0\wedge1)\vee(1\wedge0)\\ (0\wedge1)\vee(1\wedge0)\vee(0\wedge1)\vee(1\wedge1) & (0\wedge0)\vee(1\wedge1)\vee(0\wedge1)\vee(1\wedge0)\\ (1\wedge1)\vee(1\wedge0)\vee(1\wedge1)\vee(1\wedge1) & (1\wedge0)\vee(1\wedge1)\vee(1\wedge1)\vee(1\wedge0)\end{pmatrix}=\begin{pmatrix}1\vee0\vee0\vee1 & 0\vee0\vee0\vee0\\ 0\vee0\vee0\vee1 & 0\vee1\vee0\vee0\\ 1\vee0\vee1\vee1 & 0\vee1\vee1\vee\end{pmatrix}$$

7. (10) Determine whether the following argument is valid. First write the argument form symbolically, using letters for positive statements only. $\quad$ M ≡ She is math major $\quad$ C ≡ she is CS m
D ≡ She knows discrete math $\quad$ S ≡ She is sm

She is a math major or a CS major.
If she does not know discrete math, she is not a math major.
If she knows discrete math, she is smart.
She is not a CS major.
∴ She is smart.

$$\begin{array}{lll} M\vee C & \equiv \neg C\to M \\ \neg D\to\neg M & \equiv M\to D \\ D\to S & \equiv D\to S \\ \neg C & \equiv \neg C \\ \hline \therefore S & \therefore S \end{array} \equiv \begin{cases}\neg C\\ \neg C\to M\\ M\to D\\ D\to S\end{cases} \Rightarrow \begin{array}{l}\neg C\\ \neg C\to S\\ \hline S\end{array}$$

Chain syllogism
Modus ponens.

Valid

8. (6) Give an *indirect* proof of: If n is an even integer, then n+1 is odd.

Assume that n+1 is not odd.
Then n+1 is even: $n+1=2m$ for some m
$n=2m-1=2(m-1)+1$
∴ n is odd. That is, n is not even.

9. (10) Use mathematical induction to prove: $1\cdot1!+2\cdot2!+3\cdot3!+\cdots+n\cdot n!=(n+1)!-1$: $P(n)$

$P(1)$: $1\cdot1!=1=2-1=(1+1)!-1$

Assume: $P(n)$: $1\cdot1!+2\cdot2!+\cdots+n\cdot n!=(n+1)!-1$

Then $1\cdot1!+2\cdot2!+\cdots+n\cdot n!+(n+1)(n+1)!=(n+1)!-1+(n+1)(n+1)!$
$=(n+1)![1+(n+1)]-1=[(n+1)+1]!-1$ which is $P(n+1)$.

10. (8) Suppose that |A| = 4 and |B| = 10. Find the number of

a. functions f: A→B.

f = set of all 4-tuples (a_1,a_2,a_3,a_4) whose entries are from B. ∴ $\boxed{10^4}$
$=10{,}000$

b. one-to-one functions f: A→B.

f = set similar to the above except that the a_i's are distinct.

44

∴ $\boxed{10\cdot9\cdot8\cdot7}=\boxed{5040}$

11. (19) Find the number of bit strings of length 12

a. (2) that begin with 110.

b. (2) that begin with11 *and* end with 10.

c. (5) that begin with 11 *or* end with 10.

d. (4) that have exactly 4 1's.

e. (6) that have exactly 4 1's and none of the 1's is adjacent to another.

a. $110b_4 b_5 \dots b_{12}$ $\boxed{2^9} = 512$

b. $11 b_3 b_4 \dots b_9 b_{10} 10$ $\boxed{2^8} = 256$

c. No. that begin with 11: $11 b_3 b_4 \dots b_{11} b_{12} = 2^{10} = 1024$

No. that end with 10: $b_1 b_2 \dots b_9 b_{10} 10 = 2^{10} = 1024$

No. that begin with 11 and end with 10 is $2^8 = 256$ from #b.

∴ no the begin with 11 or end with 10 is $2^{10} + 2^{10} - 2^8 = \boxed{2^{11} - 2^8} = 1792$

d. Choose 4 slots from the 12 to receive 1's, the others receive 0's.

$\boxed{C(12,4) = \frac{12!}{4!8!}} = 495$

e. A bit-string ends with a zero or it does not.

(i) If it ends with a zero, then it is formed by four 10 pairs plus 4 0's. Of the 8 slots, choose 4 to receive the 10 pairs, the remainder receive 0's. $C(8,4)$

(ii) If ends with a 1, then it is formed by 3 10 pairs plus 5 0's (the last place being a 1. Of the 8 slots chose 3 to receive the 10 pairs. $C(8,3)$

$C(8,4) + C(8,3) = \boxed{C(9,4) = \frac{9!}{4!5!}} = 126$

19