(1) Compute

$$\left(\frac{1}{10}\right)^{\frac{1}{2}}\left(\frac{1}{10^2}\right)^{\frac{1}{2^2}}\left(\frac{1}{10^3}\right)^{\frac{1}{2^3}}\cdots.$$

Solution The desired infinite product can be rewritten as

$$\left(\frac{1}{10}\right)^{\frac{1}{2}}\left(\frac{1}{10}\right)^{\frac{2}{2^2}}\left(\frac{1}{10}\right)^{\frac{3}{2^3}}\cdots$$

which is the same as

$$\left(\frac{1}{10}\right)^{\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots}.$$

We first evaluate the exponent which is of the form

$$x+2x^2+3x^3+\cdots$$

with $x=\frac{1}{2}$. Now one can show that

$$\frac{1}{(1-x)^2}=1+2x+3x^2+\cdots.$$

To see that, one can start with

$$\frac{1}{1-x}=1+x+x^2+\cdots$$

and differentiate with respect to x. One can also establish this without using Calculus as

$$1+2x+3x^2+4x^3+\cdots=(1+x+x^2+\cdots)+(x+x^2+x^3+\cdots)+(x^2+x^3+x^4+\cdots)+\cdots$$

which is

$$\frac{1}{1-x}+\frac{x}{1-x}+\frac{x^2}{1-x}+\cdots=\frac{1}{1-x}(1+x+x^2+\cdots)=\frac{1}{(1-x)^2}.$$

Hence,

$$x+2x^2+3x^3+\cdots=\frac{x}{(1-x)^2}.$$

So the desired product is

$$\left(\frac{1}{10}\right)^2=\frac{1}{100}.$$

(2) Consider the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ..$, where the positive integer m appears m times. Let $d(n)$ denote the nth element of this sequence starting with $n = 1$. Find a closed-form formula for $d(n)$.

Solution Observe that $d(n) = m$ when $\frac{m(m-1)}{2} < n \leq \frac{m(m+1)}{2}$. Therefore,

$$m^2 - m < 2n \leq m^2 + m \Rightarrow m^2 - m + 1/4 < 2n < m^2 + m + 1/4$$

$\Rightarrow (m - 1/2)^2 < 2n < (m + 1/2)^2 \Rightarrow m - 1/2 < \sqrt{2n} < m + 1/2$. From this last inequality we obtain that $d(n) = \lfloor \sqrt{2n} + 1/2 \rfloor$.

(3) Let $0 < \theta < \dfrac{\pi}{2}$, prove that

$$\left(\frac{\sin^2\theta}{2} + \frac{2}{\cos^2\theta}\right)^{\frac{1}{4}} + \left(\frac{\cos^2\theta}{2} + \frac{2}{\sin^2\theta}\right)^{\frac{1}{4}} \geq (68)^{\frac{1}{4}}$$

and determine the value of θ when the inequality holds as equality.

Solution Let $u = \cos(2\,\theta)$, then $-1 < u < 1$. The left side of the inequality can be simplified to:

$$\left(\frac{1-u}{4} + \frac{4}{1+u}\right)^{\frac{1}{4}} + \left(\frac{1+u}{4} + \frac{4}{1-u}\right)^{\frac{1}{4}}$$

Apply the Arithmetic-Geometric Means Inequality, we get that:

$$\begin{aligned}
& \left(\frac{1-u}{4} + \frac{4}{1+u}\right)^{\frac{1}{4}} + \left(\frac{1+u}{4} + \frac{4}{1-u}\right)^{\frac{1}{4}} \\
\geq \quad & 2\left(\left(\frac{1-u}{4} + \frac{4}{1+u}\right)\left(\frac{1+u}{4} + \frac{4}{1-u}\right)\right)^{\frac{1}{8}} \\
= \quad & 2\left(2 + \frac{1-u^2}{16} + \frac{16}{1-u^2}\right)^{\frac{1}{8}} \\
= \quad & 2\left(2 + \frac{(1-u^2)^2 + 16^2}{16\,(1-u^2)}\right)^{\frac{1}{8}}
\end{aligned}$$

The equality holds when

$$\frac{1-u}{4} + \frac{4}{1+u} = \frac{1+u}{4} + \frac{4}{1-u}$$

which is true when $u = 0$. That is: $\cos(2\,\theta) = 0$ and so $\theta = \dfrac{\pi}{4}$.

The minimum value of

$$2\left(2 + \frac{(1-u^2)^2 + 16^2}{16\,(1-u^2)}\right)^{\frac{1}{8}}$$

occurs when $u = 0$ which is $(68)^{\frac{1}{4}}$. Thus we have

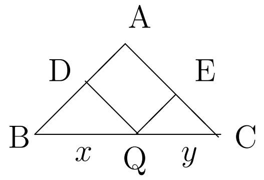
$$\left(\frac{\sin^2\theta}{2} + \frac{2}{\cos^2\theta}\right)^{\frac{1}{4}} + \left(\frac{\cos^2\theta}{2} + \frac{2}{\sin^2\theta}\right)^{\frac{1}{4}} \geq (68)^{\frac{1}{4}},$$

the equality holds when $\theta = \dfrac{\pi}{4}$. □

(4) In △ABC, parallel lines to AB and AC are drawn from a point Q lying on side BC. If a is used to represent the ratio of the area of parallelogram ADQE to the area of the triangle △ABC,

(i) find the maximum value of a.

(ii) find the ratio $\dfrac{\mathrm{BQ}}{\mathrm{QC}}$ when $a = \dfrac{24}{49}$.



Solution Let h be the height of △ABC from point A to side BC; h_x be the height from D to BQ of △BDQ; h_y be the height from E to side QC of △EQC; x be the length of BQ and y be the length QC. Then we have the following results:

$$\frac{\text{area of ADQE}}{\text{area of}\,\triangle\text{ABC}} = a,$$

$$\text{area of } \triangle\text{ABC} = \frac{1}{2}\,h(x+y),$$
$$\text{area of ADQE} = \text{area of } \triangle\text{ABC} - \frac{1}{2}(x\,h_x + y\,h_y),$$
$$\frac{h_x}{h} = \frac{x}{x+y}, \qquad \frac{h_y}{h} = \frac{y}{x+y}.$$

Thus

$$\begin{aligned}\text{Area of ADQE} = &\quad \frac{1}{2}\,h\,(x+y) - \frac{1}{2}\,h\,\left(\frac{x^2}{x+y} + \frac{y^2}{x+y}\right)\\ = &\quad \frac{h\,x\,y}{x+y}.\end{aligned}$$

So

$$\frac{\text{area of ADQE}}{\text{area of }\triangle\text{ABC}} = \frac{2\,x\,y}{(x+y)^2} = a,$$

or

$$(\frac{x}{y})^2 + 2(1-\frac{1}{a})\,\frac{x}{y} + 1 = 0.$$

By using the quadratic formula, we have

(1) $$\frac{x}{y} = \frac{1}{a} - 1 + \sqrt{\frac{1}{a}\left(\frac{1}{a}-2\right)} \quad \text{or} \quad \frac{1}{a} - 1 - \sqrt{\frac{1}{a}\left(\frac{1}{a}-2\right)}.$$

In order for the square root to exist, we require

$$\frac{1}{a}\,(\frac{1}{a}-2) \geq 0, \quad \text{or} \quad \frac{1}{a} \geq 2, \quad \text{or} \quad a \leq \frac{1}{2}.$$

So the maximum value of a is $\frac{1}{2}$.

If $a = \dfrac{24}{49}$, we get the following result from (1): $\dfrac{x}{y} = \dfrac{3}{4}$ or $\dfrac{4}{3}$. Which means that Q divides BC into a ratio of 3 to 4. □

(5) Prove the following inequality

$$\frac{1}{2009} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{2007}{2008} < \frac{1}{40}$$

Solution: $(n+1)^2 = n(n+2)+1 > n(n+2)$. So for any natural number n we have $\frac{n}{n+1} < \frac{n+1}{n+2}$ and of course $\frac{n}{n+1} > \frac{n}{n+2}$. So

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{2007}{2008} < \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdots \frac{2008}{2009}$$

and

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{2007}{2008} > \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{5}{7} \cdot \frac{7}{9} \cdots \frac{2007}{2009} = \frac{1}{2009}$$

The second inequality above justifies the desired left hand inequality. To show the desired right hand inequality, set $S = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{2007}{2008}$. Then the first inequality above gives

$$S < \frac{1}{2009S} \Rightarrow S^2 < \frac{1}{2009} < \frac{1}{1600} \Rightarrow S < \frac{1}{40}.$$