Saginaw Valley State University
2007 Math Olympics – Level I Solutions

1. A middle school has 100 lockers numbered 1 to 100, and 100 students. The first student goes down the row of lockers and opens every locker. Then the second student goes down the row of lockers and closes every locker that is numbered with a multiple of two. Then the third student goes down the row of lockers, and for every locker that is numbered with a multiple of 3, if it is open, she closes it, but if it is already closed, she opens it again. The fourth student then does the same thing for the lockers numbered with multiple of 4, and so on, down to the hundredth student. In the end, how many lockers are still open?

   (a) 1 **(b)** all of the lockers that are not numbered with prime numbers **(c)** 10

   (d) 15 **(e)** None of the above

   SOLUTION **(c)**: Consider locker number n. Notice that student m will touch the locker only if n is a multiple of m. So we want to consider all factors of n, including 1 and n. The complete list of factors represents all students who will touch the locker.

   If n has an even number of factors, then an even number of students will touch the locker. Since the first student opens it, the nth student will close it and it will remain closed. If n has an odd number of factors, then similarly we see that the nth student will open the locker and it will remain open.

   It remains to count how many n have an odd number of factors. Each factorization $n = ab$ with $a \neq b$ gives us a pair of factors, keeping the count of factors even. So the only way to get an odd number of factors is if $n = c^2$ for some c, that is if n is a perfect square.

   There are 10 perfect squares ($1^2, 2^2, \ldots, 10^2$) between 1 and 100, and so 10 lockers will remain open.

2. Which of the following equations describes the set of all points that are equidistant from the points $P(-1,3)$ and $Q(3,5)$?

   (a) $x - 2y = -7$ **(b)** $(x-1)^2 + (y-4)^2 = 5$ **(c)** $(x+1)^2 + (x-3)^2 = 5$

   (d) $2x + y = 6$ **(e)** None of the above

   SOLUTION **(d)**: We are looking for points (x, y) that have the same distance from $(-1,3)$ as from $(3,5)$. Using the distance formula, this gives us the equation

   $$\sqrt{(x+1)^2 + (y-3)^2} = \sqrt{(x-3)^2 + (y-5)^2}$$

   Squaring both sides and simplifying gives

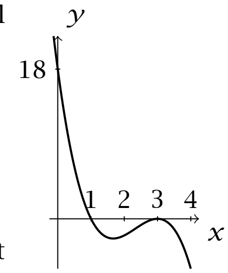
   $$\begin{aligned} x^2 + 2x + 1 + y^2 - 6y + 9 &= x^2 - 6x + 9 + y^2 - 10y + 25 \\ 2x - 6y + 10 &= -6x - 10y + 34 \\ 8x + 4y &= 24 \\ 2x + y &= 6 \end{aligned}$$

   Another solution: The described set is the perpendicular bisector of the segment $\overline{PQ}$. It must be a line perpendicular to the segment $\overline{PQ}$ passing through the midpoint of the segment. The slope of the segment $\overline{PQ}$ is

   $$m_1 = \frac{5-3}{3+1} = \frac{1}{2}$$

   and the midpoint is $\left(\frac{-1+3}{2}, \frac{3+5}{2}\right) = (1,4)$. The perpendicular bisector is the line with slope -2 passing through $(1,4)$, which has an equation $y - 4 = -2(x-1)$. Simplifying this will give us $2x + y = 6$.

3. Which of the following expressions is a factored form of the third degree polynomial function $f(x)$ whose graph is given?



(a) $18(x-1)(x-3)^2$ **(b)** $-2(x-1)(x-3)^2$

(c) $18(x-1)^2(x-3)$ **(d)** $-6(x-1)^2(x-3)$ **(e)** None of the above

SOLUTION **(b)**: The third degree polynomial has a single root at 1 and a double root at 3. Therefore it will be in the form

$$f(x) = k(x-1)(x-3)^2.$$

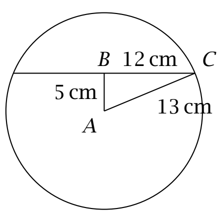
We also know that $f(0) = k(-1)(-3)^2 = -9k = 18$, which means $k = -2$.

4. Find the length of a chord that is 5 cm from the center of a circle with radius 13 cm.

(a) 12 cm **(b)** 24 cm

(c) 13 cm **(d)** Not enough information given

(e) None of the above



SOLUTION **(b)**: The triangle ABC is a right triangle with hypotenuse 13 cm and the shorter leg 5 cm. Using the Pythagorean Theorem, we obtain the other leg, which is a half of the chord. Therefore the chord is 24 cm long.

5. Find the area of a triangle (in square units) bounded by the coordinate axes and the line $x+3y-12=0$.

(a) 12 **(b)** 18 **(c)** 24 **(d)** 48 **(e)** None of the above

SOLUTION **(c)**: The x-intercept of the line is $(12,0)$ and the y-intercept is $(0,4)$. The triangle has vertices $(0,0)$, $(12,0)$ and $(0,4)$. It is a right triangle with legs 4 units and 12 units long, so its area is $\frac{1}{2} \cdot 4 \cdot 12 = 24$ square units.

6. Augustus, Benedict, Claudio, and Diana have been accused of stealing the golden mean. It is known that one of these four people must have done it. Augustus says "Benedict did it". Benedict says "Diana did it". Claudio says "I didn't do it". Diana says "Benedict is lying when he says I did it". If it is known that exactly one of them is lying, which one did it?

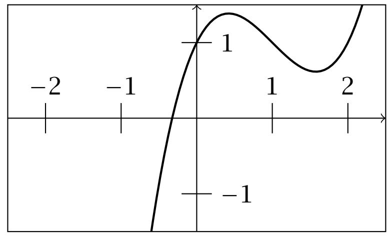
(a) Augustus **(b)** Benedict

(c) Claudio **(d)** Diana

(e) More than one person must be lying

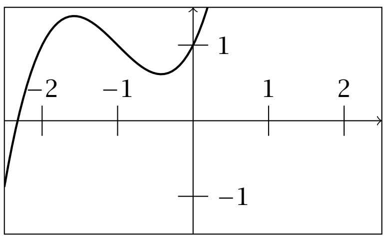
SOLUTION **(b)**: Since Benedict and Diana contradict each other, one of them must be lying. We know that only one person is lying, therefore Augustus (and Claudio, but that is not important) must be telling the truth.

7. The function f has graph

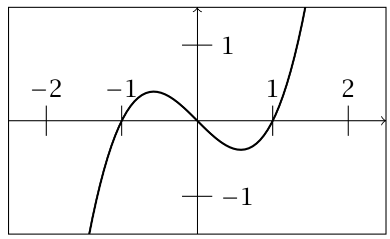


Which of the following could be the graph of $f(x+1)-1$?

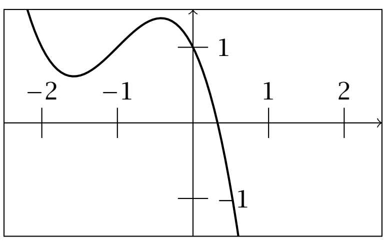
(a)



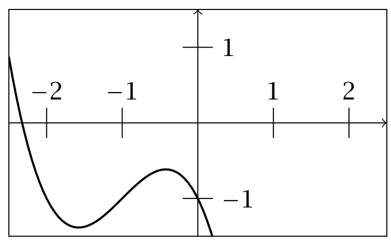
(b)



(c)



(d)



(e) None of the above

SOLUTION **(b)**: The graph of $f(x+1)-1$ is obtained from the graph of $y=f(x)$ by shifting it one unit to the left and one unit down. The only graph that could match this is **(b)**.

8. The sum of the real solutions to the equation $\left(x^3-1\right)^3-7\left(x^3-1\right)^2-4\left(x^3-1\right)+28=0$ is

(a) 7 **(b)** $\sqrt[3]{7}$ **(c)** $2+\sqrt[3]{3}$ **(d)** $1+\sqrt[3]{3}$ **(e)** None of the above

SOLUTION **(d)**: First substitute $y=x^3-1$. Factor the left side of the resulting equation by grouping:

$$\begin{aligned} y^3-7y^2-4y+28&=0 \\ y^2(y-7)-4(y-7)&=0 \\ (y^2-4)(y-7)&=0 \\ (y-2)(y+2)(y-7)&=0 \end{aligned}$$

Solving for y gives us $y=2,-2,7$. Thus

$$\begin{aligned} x^3-1&=2 & x^3-1&=-2 & x^3-1&=7 \\ x^3&=3 & x^3&=-1 & x^3&=8 \\ x&=\sqrt[3]{3} & x&=-1 & x&=2 \end{aligned}$$

The sum is $\sqrt[3]{3}-1+2=1+\sqrt[3]{3}$.

9. $\left(\frac{5}{4}-1\right)^{-\frac{1}{2}}$ is equal to

(a) 2 **(b)** $-\frac{1}{\sqrt{5}}$ **(c)** $\frac{1}{25}$ **(d)** $\frac{1}{16}$ **(e)** None of the above

SOLUTION **(a)**: $\left(\dfrac{5}{4}-1\right)^{-1/2}=\left(\dfrac{1}{4}\right)^{-1/2}=4^{1/2}=2.$

10. You can paint your living room in 6 hours and your friend would take 8 hours to do the same job. How long will it take the two of you to paint the living room if you work together?

(a) 21 minutes **(b)** $3\frac{3}{7}$ hours **(c)** 3 hours and 24 minutes

(d) $3\frac{1}{2}$ hours **(e)** None of the above

SOLUTION **(b)**: Let x be the time it takes you and your friend to paint the living room if you work together. The following table shows which fraction of the living room can be painted in one hour:

People working	**The whole room takes**	**Portion of the room painted in 1 hour**
Only you	6 hours	$\frac{1}{6}$ of the room
Only your friend	8 hours	$\frac{1}{8}$ of the room
Both together	x hours	$\frac{1}{x}$ of the room

If you work together, you paint $\frac{1}{6}$ of the room and your friend paints $\frac{1}{8}$ of the room in one hour. Therefore $\frac{1}{6}+\frac{1}{8}=\frac{1}{x}$. Multiplying both sides by the common denominator $24x$, we get $4x+3x=24$, or $x=\frac{24}{7}=3\frac{4}{7}$ hours.

11. Assume X, Y and Z represent positive real numbers. The expression $\left(\frac{16X^{-6}Y^8}{Z^{\frac{4}{3}}}\right)^{-\frac{3}{4}}$ is equivalent to

(a) $\dfrac{8X^{\frac{9}{2}}Z}{Y^6}$ **(b)** $\dfrac{ZX^4\sqrt{X}}{8Y^6}$ **(c)** $\dfrac{Y^{\frac{32}{3}}}{16X^8Z^{\frac{16}{9}}}$ **(d)** $-\dfrac{8X^{\frac{9}{2}}Y^6}{Z}$ **(e)** None of the above

SOLUTION **(b)**: Using properties of exponents:

$$\left(\frac{16X^{-6}Y^8}{Z^{\frac{4}{3}}}\right)^{-\frac{3}{4}}=\left(\left(\frac{16Y^8}{X^6Z^{\frac{4}{3}}}\right)^{\frac{3}{4}}\right)^{-1}=\left(\frac{8Y^6}{X^{\frac{9}{2}}Z}\right)^{-1}$$

$$=\frac{X^{\frac{9}{2}}Z}{8Y^6}=\frac{ZX^4\sqrt{X}}{8Y^6}$$

12. 10% of a high school senior class participate in Math Olympics. 95% of the seniors that participate in Math Olympics get into the college of their choice. Only 50% of the seniors who don't participate in Math Olympics get into the college of their choice. What percentage of seniors from that high school get into the college of their choice?

(a) 9.5% **(b)** 54.5% **(c)** 59.5% **(d)** 60% **(e)** None of the above

SOLUTION **(b)**: Out of 100 students, 10 participate in Math Olympics. Out of the 10 students, 95%, which is 9.5 students, get into the college of their choice.

Out of the 90 students who do not participate in the Math Olympics, 50%, which is 45 students, get into the college of their choice.

Altogether, out of 100 students, $9.5 + 45 = 54.5$ students get into the college of their choice.

13. How many two digit numbers are such that when the tens digit and the ones digit are interchanged, the resulting two digit number is 9 more than the original two digit number? (Note that 0 cannot be the first digit of a two digit number.)

(a) 0 **(b)** 1 **(c)** 8 **(d)** 9 **(e)** None of the above

SOLUTION **(c)**: Let d be the tens digit and e the ones digit of the number. Then the number is $10d + e$ and the number with interchanged digits is $10e + d$. Then we get

$$\begin{aligned} 10d + e + 9 &= 10e + d \\ 9d - 9e &= -9 \\ e - d &= 1 \\ e &= d + 1 \end{aligned}$$

Since d cannot be 0, the possible two digit numbers are 12, 23, 34, 45, 56, 67, 78 and 89.

14. Which of the following is an equation of a circle with diameter that has endpoints $P(2, 5)$ and $Q(6, -3)$.

(a) $(x - 4)^2 + (y - 1)^2 = 20$ **(b)** $(x - 4)^2 + (y - 1)^2 = 40$ **(c)** $(x - 2)^2 + (y - 4)^2 = 20$

(d) $(x - 2)^2 + (y - 4)^2 = 40$ **(e)** None of the above

SOLUTION **(a)**: The center of the circle must be the midpoint of P and Q, which is $\left(\frac{2+6}{2}, \frac{5-3}{2}\right) = (4, 1)$. The radius is a half of the distance between the two points:

$$r = \frac{\sqrt{(6-2)^2 + (-3-5)^2}}{2} = \frac{\sqrt{16+64}}{2} = \frac{\sqrt{80}}{2}$$

which means that $r^2 = \frac{80}{4} = 20$. The equation of the circle is then $(x - h)^2 + (y - k)^2 = r^2$, or $(x - 4)^2 + (y - 1)^2 = 20$.

15. Rationalize the denominator: $\dfrac{4\sqrt{3}}{2+\sqrt{3}+\sqrt{7}}$

(a) $\frac{4}{7}$ **(b)** $2+\sqrt{3}-\sqrt{7}$ **(c)** $-\dfrac{4\left(2-\sqrt{7}\right)}{3}$ **(d)** $2\sqrt{3}+4+\frac{4\sqrt{21}}{7}$

(e) None of the above

SOLUTION **(b)**:

$$\begin{aligned}\frac{4\sqrt{3}}{2+\sqrt{3}+\sqrt{7}}\cdot\frac{2+\sqrt{3}-\sqrt{7}}{2+\sqrt{3}-\sqrt{7}} &= \frac{\left(4\sqrt{3}\right)\left(2+\sqrt{3}-\sqrt{7}\right)}{\left(2+\sqrt{3}\right)^2-7}\\ &= \frac{\left(4\sqrt{3}\right)\left(2+\sqrt{3}-\sqrt{7}\right)}{4+4\sqrt{3}+3-7}\\ &= 2+\sqrt{3}-\sqrt{7}\end{aligned}$$

16. Find the value(s) of the parameter b, if possible, such that the linear system

$$\begin{cases} x+2y=1\\ 3x+by=3\end{cases}$$

has infinitely many solutions.

(a) $b=-2$ only **(b)** $b=6$ only

(c) Both $b=-2$ and $b=6$ **(d)** There are infinitely many such choices of b

(e) There is no such choice of b

SOLUTION **(b)**: The system will have infinitely many solutions only if one of the equations is a multiple of the other. Multiplying the first equation by 3 gives us $3x+6y=3$, which will be the same as the second equation if $b=6$.

17. To color the interior of a square of side r ft, we require exactly one can of paint. We also know we need exactly one can of paint to color the interior of a right triangle whose legs measure $(r + \frac{3}{2})$ ft and 4 ft. Assuming that the area that can be colored depends only on the amount of paint (and no other factors), what is the minimal number of cans of paint we need to buy to paint the interior of a 6 ft by 9 ft rectangle?

(a) 6 cans **(b)** 7 cans **(c)** 8 cans **(d)** 9 cans **(e)** None of the above

SOLUTION **(a)**: The square of side r ft and the right triangle with legs $(r + \frac{3}{2})$ ft and 4 ft use the same amount of paint, therefore their areas must be equal. This gives us

$$\begin{aligned} r^2 &= \frac{1}{2}\left(r + \frac{3}{2}\right) \cdot 4 \\ r^2 &= 2r + 3 \\ r^2 - 2r - 3 &= 0 \\ (r-3)(r+1) &= 0 \end{aligned}$$

therefore $r = 3$ or $r = -1$. Since r is a length, it cannot be negative, so we can discard $r = -1$. One can of paint will therefore cover exactly 9 square ft. The 6 ft by 9 ft rectangle will therefore require exactly 6 cans of paint.

18. Suppose that $f(x) = 2\left(f(x+1) + f(x-1)\right)$ for all x. If $f(2) = 2$ and $f(4) = -2$, what is $f(7)$?

(a) -1 **(b)** $\frac{3}{2}$ **(c)** $\frac{7}{4}$ **(d)** $-\frac{5}{8}$ **(e)** None of the above

SOLUTION **(c)**: First, $f(3) = 2(f(4)+f(2)) = 2(2-2) = 0$. Solving the relation $f(x) = 2\left(f(x+1) + f(x-1)\right)$ for $f(x+1)$ gives us

$$f(x+1) = \frac{1}{2}f(x) - f(x-1)$$

Therefore

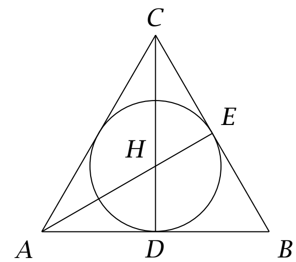
$$\begin{aligned} f(5) &= \frac{1}{2}f(4) - f(3) = -1 - 0 = -1 \\ f(6) &= \frac{1}{2}f(5) - f(4) = -\frac{1}{2} + 2 = \frac{3}{2} \\ f(7) &= \frac{1}{2}f(6) - f(5) = \frac{3}{4} + 1 = \frac{7}{4} \end{aligned}$$

19. The radius of a circle inscribed in an equilateral triangle is $2\sqrt{3}$ in. Find the exact area of the triangle.

(a) 12π in^2 **(b)** 36 in^2 **(c)** $36\sqrt{3}$ in^2 **(d)** $18\sqrt{3}$ in^2 **(e)** None of the above

SOLUTION **(c)**:

The triangle ADH is a 30-60-90 triangle, with the leg $\overline{DH}$ of length $2\sqrt{3}$ in. Therefore the other leg is $2\sqrt{3}\sqrt{3} = 6$ in long, and the hypotenuse $\overline{AH}$ is $4\sqrt{3}$ in long. The triangle CEH is congruent to ADH, therefore $CH = AH = 4\sqrt{3}$ in. Therefore the equilateral triangle ABC has base $AB = 12$ in and height $CD = 6\sqrt{3}$ in. Its area is $\frac{1}{2} \cdot 12 \cdot 6\sqrt{3} = 36\sqrt{3}$ square inches.

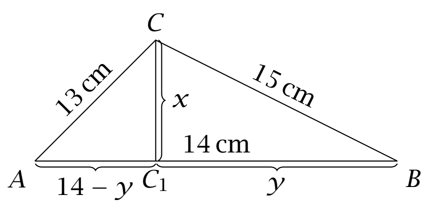


20. The lengths of the three sides of a triangle are 13 cm, 14 cm and 15 cm. Find the length of the altitude to the 14 cm side.

(a) 5 cm **(b)** 9 cm

(c) 12 cm **(d)** Not enough information given

(e) None of the above



Solution **(c)**: We need to find the length x. According to the Pythagorean theorem, $x^2 = 15^2 - y^2$ and $x^2 = 13^2 - (14 - y)^2$. This gives us the equation

$$\begin{aligned} 225 - y^2 &= 169 - 196 + 28y - y^2 \\ 225 &= 28y - 27 \\ 252 &= 28y \\ y &= 9 \end{aligned}$$

Then $x = \sqrt{15^2 - 9^2} = \sqrt{225 - 81} = \sqrt{144} = 12$ cm.

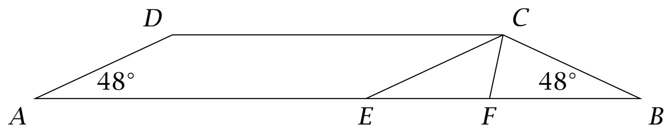
21. A farmer needs to shorten the width of a bean field by 20% to make room for a storage shed. He would like the bean field to keep the same area. By what percent should he increase the length of the field?

(a) 15% **(b)** 20% **(c)** 24% **(d)** 25% **(e)** None of the above

Solution **(d)**: Let's say the farmer needs to increase the length of the field by p%. We need to find the value of p. Let l and w be the original length and width, respectively. The original area of the field was $l \cdot w$. The new width will be $w - .2w = .8w$. The new length will be $l + \frac{p}{100}l = \left(1 + \frac{p}{100}\right)l$, and the new area will be $l(1 + \frac{p}{100}) \cdot .8w$. The farmer wants the two areas to be equal, therefore

$$\begin{aligned} .8\left(1 + \frac{p}{100}\right)lw &= lw \\ .8\left(1 + \frac{p}{100}\right) &= 1 \\ 1 + \frac{p}{100} &= 1.25 \\ \frac{p}{100} &= .25 \\ p &= 25 \end{aligned}$$

22. Given: $\overline{AB} \parallel \overline{DC}$, $\overline{CE} \parallel \overline{DA}$
$m\angle A = m\angle B = 48°$
$\overline{CF}$ bisects $\angle DCB$
Find: $m\angle FCE$



(a) 18° **(b)** 33°

(c) 42° **(d)** Not enough information given

(e) None of the above

Solution **(a)**: In the parallelogram $AECD$, the measures of $\angle A$ and $\angle DCE$ are both 48°, while the measures of $\angle D$ and $\angle E$ are both $180° - 48° = 132°$. Since $ABCD$ is an isosceles trapezoid, $m\angle DCB = m\angle D = 132°$. Since $\overline{CF}$ bisects $\angle DCB$, the measure of $\angle DCF$ is half of the measure of $\angle DCB$, or $\frac{132°}{2} = 66°$. Then $m\angle FCE = m\angle DCF - m\angle DCE = 66° - 48° = 18°$.

23. Let F be a function such that $F\left(\frac{8}{\sqrt{1+\sqrt{x}}}\right) = x$ for all $x \geq 0$. What is $F(4)$?

(a) 9 **(b)** $\frac{8\sqrt{3}}{3}$ **(c)** $\frac{-12+8\sqrt{3}}{3}$ **(d)** $\frac{9}{16}$ **(e)** None of the above

Solution **(a)**: We are looking for x such that $4 = \frac{8}{\sqrt{1+\sqrt{x}}}$. Then $F(4) = x$.

$$\begin{aligned} 4 &= \frac{8}{\sqrt{1+\sqrt{x}}} \\ \sqrt{1+\sqrt{x}} &= 2 \\ 1+\sqrt{x} &= 4 \\ \sqrt{x} &= 3 \\ x &= 9 \end{aligned}$$

Therefore $F(4) = 9$.

24. The solution to the inequality $\dfrac{x-1}{x-2} \geq 3$ is

(a) $x \leq \frac{5}{2}, x \neq 2$ **(b)** $x < 2$ or $x \geq \frac{5}{2}$ **(c)** $2 < x \leq \frac{5}{2}$ **(d)** $2 \leq x \leq \frac{5}{2}$

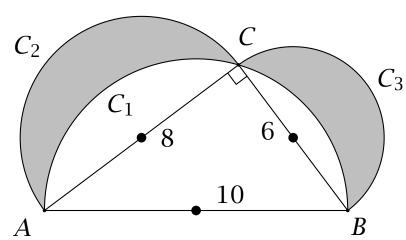
(e) None of the above

Solution **(c)**: First subtract 3 from both sides of the inequality and simplify the left side:

$$\begin{aligned} \frac{x-1}{x-2} - 3 &\geq 0 \\ \frac{x-1}{x-2} - \frac{3x-6}{x-2} &\geq 0 \\ \frac{x-1-(3x-6)}{x-2} &\geq 0 \\ \frac{5-2x}{x-2} &\geq 0 \end{aligned}$$

which means that $\frac{5-2x}{x-2}$ must be either positive or 0. The fraction on the left is 0 if $x = \frac{5}{2}$ and is undefined if $x = 2$. If $x < \frac{5}{2}$, the numerator $5 - 2x$ is positive, while for $x > \frac{5}{2}$ the numerator is negative. If $x < 2$, the denominator $x - 2$ is negative, while for $x > 2$, the denominator is positive. The fraction will only be positive if the numerator and the denominator have the same sign, which will only happen for $2 < x < \frac{5}{2}$ (when they are both negative). Since the fraction can be equal to 0, $\frac{5}{2}$ should be included.

25. A right triangle with sides 6, 8 and 10 is inscribed in a circle C_1. The legs of the right triangle are the diameters of the half circles that lie outside the triangle as shown. Find the area of the shaded region.



(a) 28

(b) 28π **(c)** 24 **(d)** 24π **(e)** None of the above

Solution **(c)**: When a right triangle is inscribed in a circle, its hypotenuse is always a diameter of the circle. Therefore the radius of the large circle is 5, while the radii of the two smaller circles are 4 and 3. The area of the shaded region is the combined area of the two small semicircles and the triangle, minus the area of the large semicircle:

$$\frac{1}{2}\pi 4^2 + \frac{1}{2}\pi 3^2 + \frac{1}{2} \cdot 6 \cdot 8 - \frac{1}{2}\pi 5^2 = \frac{1}{2}\pi(16 + 9 - 25) + 24 = 24$$

(which happens to be the area of the triangle. Would this happen for every right triangle?)