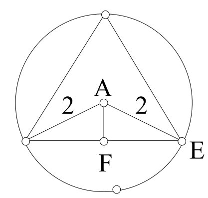
4/12/02 **2002 Math Olympics - Level I Solutions**

1) **(E)** The value of $\left(3^{-1}-2^{-1}\right)^{-1}$ is $\left(\frac{1}{3}-\frac{1}{2}\right)^{-1}=\left(-\frac{1}{6}\right)^{-1}=-6$.

2) **(B)** If l is the length, then the width is $l+1$. The perimeter is $4l+2$ and the area of 1 square foot is equal to $l(l+1)$. Solving $l(l+1)=1$, we get $l^2+l-1=0$. From the quadratic formula, the only positive root is $l=\dfrac{-1+\sqrt{5}}{2}$. The perimeter is $4l+2=4\left(\dfrac{-1+\sqrt{5}}{2}\right)+2=2\sqrt{5}$.

3) **(B)** We have $f(0)=c=1,\ f(1)=a+b+c=3,\ f(2)=4a+2b+c=2$. Since $c=1$, the last two equations are $a+b+1=3,\ 4a+2b+1=2$. Solving the first for b, $b=2-a$, and substituting into the second, $4a+2(2-a)+1=2$, the result is $2a+5=2$ or $a=-\frac{3}{2}$.

4) **(B)** The triangle AFE is a 30-60-90 triangle with AE = 2. Therefore, AF = 1 and EF = $\sqrt{3}$, so the side length is $2\sqrt{3}$.



5) **(A)** Let T be the sum of the test scores before the error was discovered and n the number of students who took the exam. This gives $T/n=75$ or $T=75n$. After the error the sum is then $T+10=75n+10$, so the average is then $\dfrac{75n+10}{n}=75.4$. Solving for n, we get $0.4n=10$ or $n=25$.

6) **(E)** There are $10\cdot10\cdot10=1000$ possible outcomes since repetition is allowed. The two digits that are the same, of which there are 10 choices could be drawn in any of three orders; the first two, the last two, or the first and last. For each choice of the repeated digit, there are 9 choices for the third, since it cannot be the same. By the multiplication rule, then, there are $10\cdot3\cdot9=270$ of the 1000 possible outcomes with precisely two the same. $270/1000=27/100$ is not one of the options.

7) **(A)** Since the triangle is inscribed in a semicircle, it is a right triangle with right angle at C. By the Pythagorean Theorem, $CD=\sqrt{16-x^2}$, so the area of the triangle is $\dfrac{1}{2}x\sqrt{16-x^2}$.

8) **(C)** If we let $x=\log_{1/9}3$, then this is equivalent to $\left(\frac{1}{9}\right)^x=3$ or $(3^{-2})^x=3$. Equating exponents, $-2x=1$ or $x=-\frac{1}{2}$.

9) **(B)** $(f\circ f)(1)=f(f(1))=f(-2)=-2(-2)^2=-8.$

10) **(A)** If we let (x, y) be the coordinate of the center, then, from the distance formula or the Pythagorean Theorem $x^2 + y^2 = 25$. And, since the point is on the line $x = 2y$, we substitute to get $4y^2 + y^2 = 25$, or $y = \sqrt{5}$ since it is in the first quadrant. Therefore, $x = 2y = 2\sqrt{5}$.

11) **(D)** $\left(\dfrac{-8x^3}{y^{-6}}\right)^{-1/3} = \left(-8x^3y^6\right)^{-1/3} = \left(\dfrac{1}{-8x^3y^6}\right)^{1/3} = \dfrac{-1}{2xy^2}$.

12) **(C)** If d is the distance (in miles) she runs north, then d is also the distance she runs south. The total time is then $d/10 + d/12 = 44/60$ after converting the 44 minutes to hours. Solving, we get $11d/60 = 44/60$, so $d = 4$ and the total mileage is 8 miles.

13) **(B)** Notice that $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32, 2^6 = 64, 2^7 = 128, 2^8 = 256, \cdots$ so the last digits follow the pattern $2,4,8,6,2,4,8,6,\cdots$. The last digit is a 6 if the exponent is a multiple of 4. Dividing 2002 by 4, then, we get a remainder of 2, so the last digit is 4 (two past the 6 in the pattern).

14) **(C)** As each triangle is a 45-45-90 triangle, $AD = BC = 3\sqrt{2}$, and the hypotenuse of the triangle with BC as one leg is $3\sqrt{2}\cdot\sqrt{2} = 6$. Since B is the midpoint we know the dimensions of the larger rectangle are 9 and 12 and $AB = 6\sqrt{2}$, Therefore, the area is $3\sqrt{2}\cdot 6\sqrt{2} = 36$.

15) **(B)** Let A = Audrey's age, B = Beatrice's age, and C = Clement's age.

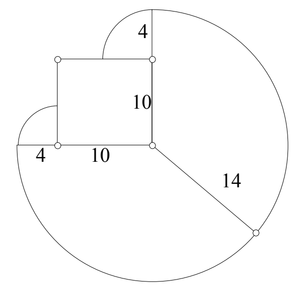
I. If (1) is false then the other three must be true, so A < C < B from (2) and (4), but then it is impossible to have B + C = 2A from (3). Therefore, (1) must be true.
II. If (2) is false, we have C > A > B from (1) and (4) being true. Now it is possible to have B + C =2A, so there is no contradiction.
III. If (3) is false, then (1) and (2) must be true, so A > B > C . But (4) must also be true, so C > A, which is a contradiction.
IV. If (4) is false, then (1) and (2) must be true, so A > B > C. But (3) being true implies B + C =2A, which is impossible.

Therefore, (2) must be false and Beatrice is the youngest.

16) **(D)** Since these are prime numbers, the unit's digit cannot be 4, 6, or 8, so these must be the ten's digit. Since 63 and 69 are both divisible by three, 61 must be one of the prime numbers. This leaves 43 or 49 with 4 in the ten's place, and 43 is prime. This leaves 89 for the third prime. We have 89 > 61 > 43, so p = 89, q = 61, and r = 43. Now, p - q + r = 89 - 61 + 43 = 71.

17) **(A)** The image of the point (a, b), when reflected across the y - axis, is $(-a, b) = (c, d)$. The image of $(-a, b)$, when reflected across the x -axis, is $(-a, -b) = (e, f)$. So, $ab - ef = ab - (-a)(-b) = 0$.

18) **(C)** From the figure, we see that there is three-fourth's of a circle of radius 14 and two quarter-circles of radius 4. So the total area is $\frac{3}{4}\pi(14)^2 + 2\left(\frac{1}{4}\pi(4)^2\right) = 147\pi + 8\pi = 155\pi$.



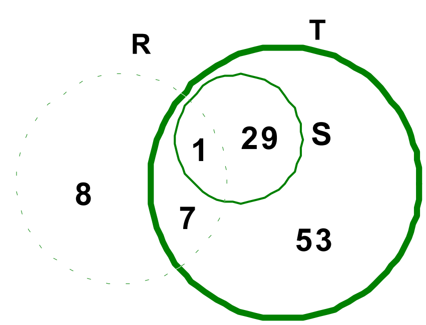
19) **(D)** If $b^a = 1$ then one of the following three cases must be true.

I. $a = 0 \quad and \quad b \neq 0$: Solving $x^2 - 9x + 20 = 0$, $(x-4)(x-1) = 0$ or $x = 4, 5$. Checking, $x^2 - 5x + 5 \neq 0$ at 4 or 5.

II. $b = 1$: Solving $x^2 - 5x + 5 = 1 \Leftrightarrow x^2 - 5x + 4 = 0$, so $(x-4)(x-1) = 0$ or $x = 1, 4$.
III. $b = -1 \quad and \quad a \quad is\ even$: Solving,
$x^2 - 5x + 5 = -1 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow (x-6)(x+1) = 0$,
so $x = -1, 6$. Check that
$(-1)^2 - 9(-1) + 20 = 30,\ 6^2 - 9(6) + 20 = 2$ are both even.

Summing all the different values, $4 + 5 + 1 + (-1) + 6 = 15$.

20) **(B)** There are 90 Tufs, and since one-third are Sufs, there are 30 Sufs. All the Sufs are Tufs, so there are 60 Tufs that are not Sufs. Eight of those Tufs are Rufs and one of those eight is also a Suf, so there are 7 Rufs that are Tufs and not Sufs. This leaves 53 Tufs that are neither Sufs or Rufs. (The Venn Diagram is a little easier to follow.)



21) **(C)** The distance from a point to a line is measured along the perpendicular line, so we first find the equation of the line perpendicular to $y = \frac{1}{2}x + 5$ that goes through the point $(8, -1)$. This line must have slope $m=-2$ (the negative of the reciprocal of $1/2$), so its equation is $y+1=-2(x-8)$ or $y=-2x+15$.

To find the intersection of the two lines, we solve $\frac{1}{2}x+5=-2x+15$, which results in $x=4$. The y coordinate is then 7, so the distance to the line is the distance from $(8,-1)$ to $(4,7)$, or $\sqrt{4^2+8^2}=\sqrt{80}=4\sqrt{5}$.

22) **(C)** Solving the first equation for x^2, we get $x^2=-2y+4$, and, substituting into the second equation, $y^2-2y+4=4$ or $y^2-2y=0$. This gives $y=2,0$. Substituting into either equation, if $y=2$ then $x=0$, and if $y=0$ then $x=\pm 2$. Therefore, there are three points of intersection, $(0,2), (2,0), (-2,0)$.

23) **(D)** If A is the cost of an apple and O the cost of an orange (in cents) , then $3A+2O=84$ and $6A+O=132$. Multiplying the first equation by -2 we have $-6A-4O=-168$, and adding this equation to the second the result is $-3O=-36$ or $O=12$. Substituting into the second equation, $6A+12=132$ or $A=20$. Therefore, $2A+2O=2(20)+2(12)=64$.

24) **(C)** The average speed is the total distance divided by total time. Her total time is 5 hours and the total distance is $50\cdot 2+70\cdot 3=310$. Therefore, the average speed is $\frac{310}{5}=62$ mph.

25) **(C)** If square has side length s and the circle has radius r then the perimeters are $4s$ and $2\pi r$, respectively. Since these are equal, $4s=2\pi r$, and $\frac{s}{r}=\frac{2\pi}{4}=\frac{\pi}{2}$.