Saginaw Valley State University
2017 Math Olympics — Level II

1. The power set of a set S is the set of all the subsets of S. The empty set is denoted $\emptyset$. Which of the following is a power set of some set?

   (a) $W = \{\{a\}, \{\{b\}\}, \{a, \{b\}\}\}$ **(b)** $X = \{\emptyset, \{a\}, \{b\}, \{a, \{b\}\}\}$

   (c) $Y = \{\emptyset, \{a\}, \{\{b\}\}, \{a, \{b\}\}\}$ **(d)** $Z = \{\emptyset, \{a\}, \{\{b\}\}, \{a, b\}\}$

   (e) $A = \{\emptyset, \{a\}, \{a, b\}\}$

2. At one of those leadership conferences, there is one of those icebreakers where each of the 50 participants is labeled with a distinct integer between 1 and 50, inclusive. Each odd-numbered participant must shake hands exactly once with each of the other participants, and no even numbered participants may shake hands with each other. How many handshakes take place?

   (a) 450 **(b)** 925 **(c)** 950 **(d)** 1225 **(e)** None of the above

3. Find

$$\sum_{k=1}^{2017} i^k$$

   where $i^2 = -1$.

   (a) -1 **(b)** i **(c)** 1 **(d)** $-i$ **(e)** $1 + i$

4. How many integers n with $1000 \le n \le 9999$ have 4 distinct digits in increasing order or decreasing order?

   (a) 336 **(b)** 612 **(c)** 720 **(d)** 904 **(e)** None of the above

5. Johnny, Dee Dee, Joey, Tommy, and Marky are in a band. The sums of the ages of each group of four of them are 132, 138, 113, 131, and 126. What is the age of the oldest of the band members?

   (a) 39 **(b)** 43 **(c)** 45 **(d)** 47 **(e)** 49

6. For which of the following values of θ is $2^{\cos\theta} > 1$ and $3^{\sin\theta} < 1$?

   I. $\frac{5\pi}{6}$ II. $\frac{11\pi}{6}$ III. $\frac{13\pi}{8}$

   (a) I only **(b)** II only **(c)** III only **(d)** I and II only

   (e) II and III only

7. $\sin\frac{\pi}{8} + \cos\frac{\pi}{8} =$

   (a) $\frac{\sqrt{2}}{4}$ **(b)** $\frac{2+\sqrt{2}}{2}$ **(c)** $\sqrt{\frac{2-\sqrt{2}}{4}}$ **(d)** $\sqrt{1+\frac{1}{\sqrt{2}}}$ **(e)** None of the above

8. Determine the number of positive divisors of 18,800 that are divisible by 235.

   (a) 8 **(b)** 10 **(c)** 12 **(d)** 14 **(e)** 22

9. A function f is defined so that if n is an odd integer, then $f(n) = n - 1$ and if n is an even integer, then $f(n) = n^2 - 1$. For example, if $n = 15$, then $f(n) = 14$ and if $n = -6$, then $f(n) = 35$, since 15 is an odd integer and -6 is an even integer. Determine the expression of $f(f(n))$.

   (a) $f(f(n)) = n^2 - 2$ if n is even, $f(f(n)) = n^2 - 2n$ if n is odd

   (b) $f(f(n)) = n^2 - 2n$ if n is even, $f(f(n)) = n^2 - 2$ if n is odd

   (c) $f(f(n)) = 2n^2 - 1$ for all n **(d)** $f(f(n)) = 2n^2 - 2$ for all n **(e)** None of the above

10. If the operation $*$ is defined for all positive real numbers x and y by $x * y = \frac{x+y}{xy}$, which of the following must be true for positive x, y, and z?

    I. $x * x = \frac{2}{x}$ II. $x * y = y * x$ III. $x * (y * z) = (x * y) * z$

    (a) I only **(b)** I and II only **(c)** I and III only **(d)** II and III only

    (e) all three

11. What is the area of the region of the plane determined by the inequality

$$7 \le |x| + |y| \le 13?$$

    (a) 169 **(b)** 81 **(c)** 240 **(d)** 120 **(e)** 78

12. $2^{7/6} - 2^{2/3} =$

(a) $\sqrt[3]{4}(\sqrt{2}+1)$ **(b)** $\sqrt{2}$ **(c)** $\sqrt{8}(\sqrt{2}-1)$

(d) $\frac{\sqrt[3]{4}}{\sqrt{2}+1}$ **(e)** None of the above

13. If $\ln(\sec\theta - \tan\theta) = x$ then which of the following is true:

(a) $\sec\theta = \frac{e^x+e^{-x}}{2}$ **(b)** $\csc\theta = e^x$ **(c)** $\sec\theta + \tan\theta = e^x - e^{-x}$

(d) $\cos\theta = e^x - e^{-x}$ **(e)** None of the above

14. If $f(x - \frac{1}{2}) = \frac{1}{2}f(x) + 3$ and $f(0) = 0$, what is $f(2)$?

(a) $\frac{45}{8}$ **(b)** $\frac{9}{2}$ **(c)** -90 **(d)** -18 **(e)** None of the above

15. $\cos(2\sin^{-1}(x))$ is equivalent to which of the following?

(a) $2\sqrt{1-x^2}$ **(b)** $2x\sqrt{1-x^2}$ **(c)** $\sqrt{1-x^2}$

(d) $1 - 2x^2$, $|x| \le 1$ **(e)** None of the above

16. What condition on c will guarantee that the equation $e^x + ce^{-x} = 2$ has two solutions?

(a) $c < 1$ **(b)** $0 < c < 1$ **(c)** $c < 0$ **(d)** $c > 1$ **(e)** $c > 0$

17. Write

$$\frac{(-1)^{2017} + i^{2017}}{5 - \sqrt{-5}}$$

as $a + bi$ where a and b are real numbers.

(a) $\frac{-5-\sqrt{5}}{20} + \frac{5-\sqrt{5}}{20}i$ **(b)** $\frac{-5+\sqrt{5}}{20} - \frac{5-\sqrt{5}}{20}i$ **(c)** $\frac{-5-\sqrt{5}}{30} + \frac{5-\sqrt{5}}{30}i$

(d) $\frac{-5+\sqrt{5}}{30} - \frac{5-\sqrt{5}}{30}i$ **(e)** None of the above

18. If $6\sec^2\theta + 7\tan\theta - 16 = 0$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, what are possible values of $\sin\theta$?

(a) $\frac{5\sqrt{61}}{61}$ and $1/2$ **(b)** $1/3$ and $-\frac{2\sqrt{5}}{5}$ **(c)** $\frac{5\sqrt{61}}{61}$ and $-\frac{2\sqrt{5}}{5}$

(d) $1/3$ and $1/6$ **(e)** None of the above

19. Find the sum of the integers in the arithmentic sequence

$$11, 28, 45, 62, \ldots, 2017$$

(a) 119,652 **(b)** 119,663 **(c)** 120,655 **(d)** 120,666 **(e)** None of the above

20. Let $f(x) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)x$ for a complex number x. Define $f^{(n)}(x)$ as the n-times composition of f, i.e. $f^{(1)}(x) = f(x)$ and $f^{(n)}(x) = f\left(f^{(n-1)}(x)\right)$. What is $f^{(2017)}(x)$?

(a) $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)x$ **(b)** x **(c)** $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)x$

(d) $-x$ **(e)** None of the above

21. In a drawer, Isaac has 10 socks, with two of each of 5 different colors. On Monday, Isaac selects two individual socks at random from the 10 socks in the drawer. On Tuesday, Isaac selects 2 of the remaining 8 socks at random and on Wednesday two of the remaining 6 socks at random. What is the probability that Wednesday is the first day Isaac selects 2 socks of the same color?

(a) $26/315$ **(b)** $8/9$ **(c)** $24/53$ **(d)** $10/189$ **(e)** None of the above

22. Which of the following statements are true?

I. For every real number x, there exists a real number y such that $x + y = 0$.

II. There exists a real number x such that for every real number y, $x + y = 0$.

III. There exists a real number x such that for every real number y, $xy = y$.

(a) I only **(b)** II only **(c)** III only **(d)** I and II only

(e) I and III only

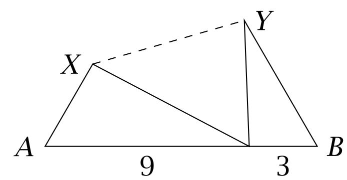
23. Given that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$, find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}$$

(a) $27^2 + 49^2 + 11$ **(b)** $54 + 98 + 2\sqrt{11}$ **(c)** $7^3 + 11^2 + 5$

(d) $7^3 + 11^7 + 25^3$ **(e)** None of the above

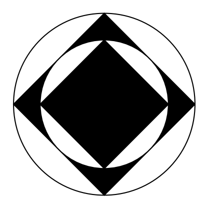
24. An equilateral triangle $\triangle ABC$ with side length 12 is folded in such a way that the vertex C touches the side $\overline{AB}$ at a point that is 9 units away from A and 3 units away from B, as shown. Find the length XY.



(a) $\frac{39\sqrt{39}}{35}$ **(b)** $\frac{78\sqrt{6}}{35}$ **(c)** $3\sqrt{7}$ **(d)** $4\sqrt{3}$

(e) None of the above

25. Kenny and Meghen found a cool dart board at a garage sale. It was a white circle in which there was inscribed a black square, in which there was an inscribed white circle, in which there was inscribed a black square, as shown. They were having a debate whether there was more black than white on the dartboard. Meghen said there was more white but Kenny was sure the black area was much larger. He thought that the black area was at least 1.5 times larger than the white area. Rich, who was also shopping at the same garage sale, thought that the black area was larger than the white area, but less than 1.5 times larger. The seller claimed the two areas were equal. Which one of them was right?



(a) Meghen was right, the white area is larger.

(b) The seller was right, the two areas are equal.

(c) Rich was right, the black area is larger, but less than 1.5 times larger.

(d) Kenny was right, the black area is at least 1.5 times larger than the white area.

(e) It is impossible to determine without knowing the diameter of the large circle.