Saginaw Valley State University

2016 Math Olympics — Level I Solutions

1. Suppose $\frac{a}{b}$ is a fraction in the simplest form, where $0 < a < b$, and $b = 2^8 5^5$.
How many decimal places are there to the right of the decimal point in the decimal expansion of $\frac{a}{b}$?

(a) 5 **(b)** 8 **(c)** 13

(d) Not enough information **(e)** None of the above

SOLUTION **(b)**: We know that a is less than b and is divisible by neither 2 nor 5 (the fraction is in the simplest form). Multiplying both numerator and denominator by 5^3 gives us

$$\frac{a}{b} = \frac{a \cdot 5^3}{2^8 \cdot 5^5 \cdot 5^3} = \frac{a \cdot 5^3}{2^8 \cdot 5^8} = \frac{a \cdot 5^3}{10^8}$$

where the numerator $a \cdot 5^3$ is less than 10^8 and is not divisible by 10. Therefore the numerator has a non-zero digit at the ones place, and the decimal expansion of $\frac{a}{b}$ has a non-zero digit 8 places to the right of the decimal point.

2. Given that

$$x + y = -5$$
$$x^2 + y^2 = 73$$

find $x^3 + y^3$.

(a) -125 **(b)** -245 **(c)** -365

(d) -485 **(e)** Not enough information

SOLUTION **(d)**: According to the sum of cubes formula, $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$. Since we already know $x + y$ and $x^2 + y^2$, all wee need to find is xy. But since $(x+y)^2 = x^2 + 2xy + y^2$, we get $xy = \frac{1}{2}\left((x+y)^2 - (x^2+y^2)\right) = \frac{1}{2}\left((-5)^2 - 73\right) = -24$. Then

$$x^3 + y^3 = (x+y)(x^2+y^2-xy) = (-5)(73-(-24)) = (-5)(97) = -485$$

3. How many rational roots does

$$x^{1000} - x^{500} + x^{100} + x + 1 = 0$$

have?

(a) 0 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** None of the above

SOLUTION **(a)**: According to the rational root theorem, the only rational numbers that could be roots of this equation are 1 and -1. Plugging them in shows that none of them works. There are no rational roots.

4. In 2015, one hundred students participated in a math competition, and 20% of them scored at least 20 points. In 2016, one hundred twenty students participated in the competition, and 25% of them scored at least 20 points. What was the percent increase in the number of students who scored at least 20 points in the competition between 2015 and 2016?

(a) 0% **(b)** 5% **(c)** 20% **(d)** 50% **(e)** None of the above

SOLUTION **(d)**: In 2015, the number of students who scored at least 20 points in the competition was $.2 \cdot 100 = 20$. In 2016, there were $.25 \cdot 120 = 30$ students that scored at least 20 points. That is 10 more students, and 10 is 50% of 20.

5. Which of the following is a factor of $t^2 - u^2 + 4t + 2u + 3$?

(a) $2t + u$ **(b)** $t + u + 1$

(c) $t + u + 3$ **(d)** The polynomial does not factor

(e) None of the above

SOLUTION **(b)**:

$$\begin{aligned} t^2 - u^2 + 4t + 2u + 3 &= t^2 + 4t + 4 - u^2 + 2u - 1 \\ &= (t^2 + 4t + 4) - (u^2 - 2u + 1) \\ &= (t + 2)^2 - (u - 1)^2 \\ &= [(t + 2) + (u - 1)]\,[(t + 2) - (u - 1)] \\ &= [t + u + 1]\,[t - u + 3] \end{aligned}$$

6. Which of the following is equal to the sum

$$\frac{2}{3} + \frac{2}{8} + \cdots + \frac{2}{k^2-1} + \cdots + \frac{2}{2016^2-1}$$

(a) $\dfrac{2\left((2016^2-1)! + (2015^2-1)! + \cdots + 15! + 8! + 3\right)}{(2016^2-1)!}$

(b) $\dfrac{7 \cdot 2016 - 10}{2016 \cdot 2017}$

(c) $\dfrac{3}{2} - \dfrac{4033}{2016 \cdot 2017}$

(d) $\frac{2015}{2016}$

(e) None of the above

Solution **(c)**: Since

$$\frac{2}{k^2-1} = \frac{2}{(k-1)(k+1)} = \frac{1}{k-1} - \frac{1}{k+1},$$

the sum can be written as

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots + \frac{1}{2014} - \frac{1}{2016} + \frac{1}{2015} - \frac{1}{2017}.$$

Most fraction appear twice, one with + sign and once with − sign, so they cancel each other. All that is left is

$$1 + \frac{1}{2} - \frac{1}{2016} - \frac{1}{2017}$$

which simplifies to

$$\frac{3}{2} - \frac{4033}{2016 \cdot 2017}.$$

7. $\sqrt{\dfrac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}} =$

(a) $\sqrt{2}+1$ **(b)** $\sqrt{2}-1$ **(c)** $\frac{1}{\sqrt{2}}-1$ **(d)** $\frac{1}{\sqrt{2}}+1$ **(e)** None of the above

SOLUTION **(a)**:

$$\begin{aligned}\sqrt{\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}} &= \sqrt{\frac{\left(1+\frac{1}{\sqrt{2}}\right)\sqrt{2}}{\left(1-\frac{1}{\sqrt{2}}\right)\sqrt{2}}} = \sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}} \\ &= \sqrt{\frac{\left(\sqrt{2}+1\right)\left(\sqrt{2}+1\right)}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}} = \sqrt{\frac{\left(\sqrt{2}+1\right)^2}{2-1}} \\ &= \frac{\sqrt{\left(\sqrt{2}+1\right)^2}}{1} = \sqrt{2}+1\end{aligned}$$

8. George walks x meters in twenty minutes. If his rate remains constant how many more meters does he walk in the next thirteen minutes?

(a) $\frac{20x}{13}$ meters **(b)** $\frac{x}{20}+13$ meters **(c)** $\frac{20}{x}+13$ meters **(d)** $\frac{x+13}{20}$ meters

(e) $\frac{13x}{20}$ meters

SOLUTION **(e)**: distance = rate × time, so his rate is given by $x = \text{rate} \times 20$. So his rate is $\frac{x}{20}$ meters/minute. So in the next thirteen minutes he will walk $\frac{x}{20}$ meters/minute × 13 minutes $= \frac{13x}{20}$ meters.

9. Line 1 has intercepts $(2,0)$ and $(0,-3)$. Line 2 goes through the origin and has slope 1. What is the x coordinate of their point of intersection?

(a) 3 **(b)** 6

(c) $-\frac{6}{5}$ **(d)** The lines don't intersect.

(e) None of the above

SOLUTION **(b)**: The slope of the first line is $\frac{0-(-3)}{2-0} = \frac{3}{2}$. So the equation of this line is $y = \frac{3}{2}x - 3$. The equation of the other line is $y = x$. Substituting $y = x$ into the first line gives the equation $x = \frac{3}{2}x - 3$. Solving for x: $2x = 3x - 6$ or $-x = -6$ or $x = 6$.

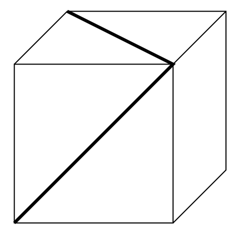
10. On which of the following intervals is it true that $|x^2 - 1| + |x^2 - 9|$ is always equal to 8?

(a) $-3 < x < 3$ **(b)** $-3 < x < -1$ **(c)** $-1 < x < 1$ **(d)** $-\infty < x < -3$

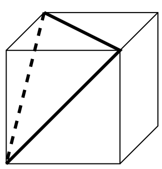
(e) $x > 1$

Solution **(b)**: If x^2-1 and x^2-9 are both positive, $|x^2 - 1|+|x^2 - 9| = x^2-1+x^2-9 = 2x^2-10$. If $x^2 - 1$ and $x^2 - 9$ are both negative $|x^2 - 1| + |x^2 - 9| = -(x^2 - 1) - (x^2 - 9) = -2x^2 + 10$. If $x^2 - 1$ is positive and $x^2 - 9$ is negative, $|x^2 - 1| + |x^2 - 9| = x^2 - 1 - (x^2 - 9) = -1 + 9 = 8$. $x^2 - 1$ is positive when $x < -1$ or $x > 1$, $x^2 - 9$ is negative when $-3 < x < 3$. So to have $x^2 - 1$ is positive and $x^2 - 9$ is negative, we must have $-3 < x < -1$ or $1 < x < 3$.

11. Shown is a cube and a diagonal on the top face of the cube and a diagonal on the front face of the cube. What is the acute angle between the two diagonals shown?



(a) 45° **(b)** 60° **(c)** 75° **(d)** 90° **(e)** None of the above



Solution **(b)**: The two diagonals, together with a diagonal on the left face, as shown on the left, form a triangle. Since all the faces of the cube are squares with the same dimensions, all the diagonals have the same length, and so they form an equilateral triangle. All three angles in an equilateral triangle measure 60°.

12. Which of the following is equivalent to $\sqrt{1\%}$?

(a) 0.1% **(b)** 1% **(c)** 1.1% **(d)** 10% **(e)** 100%

Solution **(d)**: $\sqrt{1\%} = \sqrt{0.01} = \sqrt{\frac{1}{100}} = \frac{1}{\sqrt{100}} = \frac{1}{10} = \frac{10}{100} = 10\%$.

13. What is the ones digit of 2^{2016}?

(a) 0 **(b)** 2 **(c)** 4 **(d)** 6 **(e)** 8

Solution **(d)**: Since 2016 is a multiple of 4, $2^{2016} = 16^k$ for some ineteger k. All powers of 16 have 6 as the ones digit, as can easily be proved by induction.

14. What is the acute angle formed by the hour and the minute hands of a correctly working analog clock at 7:24 pm?

(a) 66° **(b)** 72° **(c)** 78° **(d)** 82° **(e)** None of the above

Solution **(c)**: Each minute, the minute hand moves $(360/60)^\circ = 6^\circ$ clockwise, and the hour hand moves $(360/720)^\circ = (1/2)^\circ$. At 7:24 pm, the hour hand moved $(7 \cdot 60 + 24) \cdot (1/2)^\circ = 222^\circ$ from its top position while the minute hand moved $24 \cdot 6^\circ = 144^\circ$ from its top position. The angle between the two hands is $222^\circ - 144^\circ - 78^\circ$.

15. When we were preparing this test, we lost the question that went with these answers. Fortunately, since there must be exactly one correct answer for any question, you can decide which one answer below is true.

    (a) The cow jumped over the moon.

    (b) The cow jumped over the moon, or the little dog laughed to see such a sight.

    (c) The little dog laughed to see such a sight, or the dish ran away with the spoon.

    (d) The dish ran away with the spoon and the cow jumped over the moon.

    (e) If the cow didn't jump over the moon, then the little dog laughed to see such a sight.

    SOLUTION **(c)**: We know that exactly one of the above statements is true. It cannot be **(a)**, since if **(a)** is true, **(b)** and **(e)** will be true as well, and only one of the statements can be true.
    If **(b)** was true, then **(a)** or **(c)** would be true as well. Again, that is not possible, since only one of the statements can be true.
    If **(d)** was true, then **(a)** would be true as well, which cannot happen.
    If **(e)** was true, then, since we already know that **(a)** is false, "the little dog laughed to see such a sight" would have to be true, and **(c)** would be true as well. So **(e)** cannot be true.
    We ruled out **(a)**, **(b)**, **(d)** and **(e)**. Since one of them must be true, it has to be **(c)**.

16. Suppose $-1 < x < 0$. Which of the following is always true?

    (a) $x^3 < x < x^4 < x^2$ **(b)** $x^3 < x < x^2 < x^4$ **(c)** $x < x^3 < x^2 < x^4$

    (d) $x^4 < x^3 < x^2 < x$ **(e)** None of the above

    SOLUTION **(e)**: Since $x < 0$, the odd powers will be negative and the even powers will be positive. Since $|x| < 1$, the higher powers will be closer to 0 then the lower powers. That means $x < x^3 < x^4 < x^2$. None of the given choices is correct.

17. Two rhinos are charging directly at each other at constant speeds of v and w feet per second, respectively. You foolishly approach them and start your stopwatch when they are d feet apart. Which expression gives the number of seconds before they collide?

    (a) $\frac{d}{v} + \frac{d}{w}$ **(b)** $\frac{v+w}{d}$ **(c)** $\frac{d}{v+w}$ **(d)** $\frac{d}{2}v^2 + w^2$ **(e)** None of the above

    SOLUTION **(c)**: Every second the first rhino moves v feet closer to the collision point, while the second rhino moves w feet closer to the collision point. Since they are charging directly at each other, the distance between them decreases $v + w$ feet every second. It will take $\frac{d}{v+w}$ second for the distance to decrease to 0, at which moment the collision will occur.

18. Which of the following statements are true for all real numbers x?

I. $\sqrt{x^2} = x$ III. $\sqrt{x^4} = x^2$

II. $\sqrt[3]{x^3} = x$ IV. $\sqrt{x^6} = x^3$

(a) I only **(b)** I and III only **(c)** II only **(d)** II and III only

(e) I, II, III and IV

Solution **(d)**: The statements I and IV are false when x is negative. For example: $\sqrt{(-3)^2} = \sqrt{9} = 3 \neq -3$ and $\sqrt{(-2)^6} = \sqrt{64} = 8 \neq (-2)^3$.

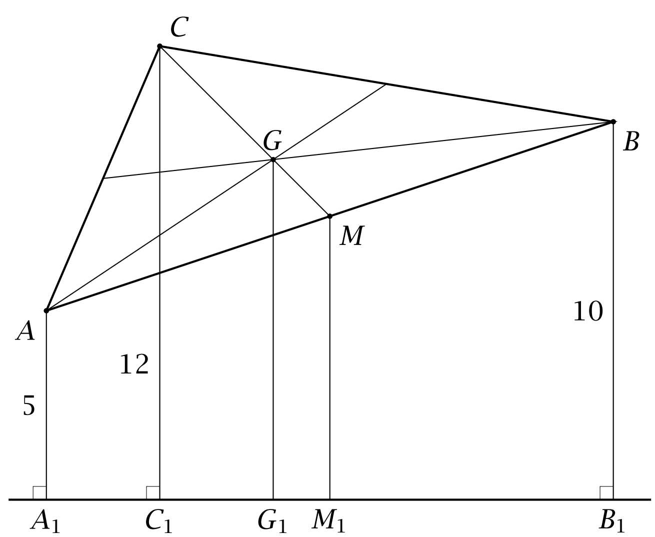
The statements II and III are true for all real numbers.

19. Given a triangle $\triangle ABC$, let G be the centroid of the triangle and l be a line not intersecting any of the sides of the triangle. If the distances of the vertices A, B and C to the line l are 5, 10 and 12 inches, respectively, what is the distance form G to l?

(a) 7 inches **(b)** 9 inches **(c)** 11.5 inches

(d) 13.5 inches **(e)** None of the above

Solution **(b)**: On the picture below, M is the midpoint of AB and $MG : GC = 1 : 2$.



Since $AA_1 \parallel BB_1$, $MM_1 = \frac{5+10}{2} = \frac{15}{2}$.

Since $CC_1 \parallel MM_1$, $GG_1 = \frac{1}{3}CC_1 + \frac{2}{3}MM_1 = \frac{1}{3} \cdot 12 + \frac{2}{3}\frac{15}{2} = 9$.

20. Ma Pickle is used to measuring vinegar for her pickle recipe from her blue coffee cup. She uses one and a half blue coffee-cups worth of vinegar. One day she breaks her blue coffee-cup and has to use her red coffee-cup instead. The red coffee-cup is an exact scale model of the blue coffee cup. The blue cup was 5 inches tall and the red cup is 3 inches tall. How many red coffee-cups worth of vinegar will equal one and a half blue coffee-cups of vinegar? (We assume Ma Pickle is good at estimating fractions of cups worth of liquid.)

(a) 9/10 **(b)** 5/2 **(c)** 125/18 **(d)** 250/81 **(e)** None of the above

SOLUTION **(c)**: The linear scaling factor between the blue and the red cup is 5/3 since the height of the blue cup is (5/3) times the height of the red cup. That means the volume of the blue cup is $(5/3)^3$ times the volume of the red cup. So each blue cup worth of volume is the same as $(5/3)^3$ red cups worth of volume. So 3/2 blue cups is $3/2 \times (125/27) = 125/18$.

21. $i^{1887} + \frac{3-i}{1-i} =$

(a) 2 **(b)** 1 **(c)** $2+i$ **(d)** $1-3i$ **(e)** None of the above

SOLUTION **(a)**:

$$\begin{aligned}
i^{1887} + \frac{3-i}{1-i} &= i^{1886}i + \frac{3-i}{1-i} \\
&= \left(i^2\right)^{943} i + \frac{3-i}{1-i} \\
&= (-1)^{943}\, i + \frac{3-i}{1-i} \\
&= -i + \frac{3-i}{1-i} \\
&= -i + \frac{(3-i)(1+i)}{(1-i)(1+i)} \\
&= -i + \frac{3+3i-i-i^2}{1-i^2} \\
&= -i + \frac{3+3i-i+1}{1+1} \\
&= -i + \frac{4+2i}{2} = -i + 2 + i = 2
\end{aligned}$$

22. If $f(x) = ax^2 + bx + c$, find b so that the graph of $y = f(x)$ passes through the points $(3, -4)$, $(-1, 12)$ and $(2, -3)$.

    (a) 6 **(b)** -6 **(c)** 12 **(d)** -12 **(e)** None of the above

    SOLUTION **(b)**: The graph of $y = f(x)$ passing through the three points gives us a system of three equations:

$$\begin{aligned} 9a + 3b + c &= -4 \\ a - b + c &= 12 \\ 4a + 2b + c &= -3 \end{aligned}$$

    If we subtract the second equation from the first we get: $8a + 4b = -16$ Subtract the second equation from the 3rd: $3a + 3b = -15$. Dividing the first of these by 4 and the second by 3 gives us the system:

$$\begin{aligned} 2a + b &= -4 \\ a + b &= -5 \end{aligned}$$

    Subtracting the second from the first gives $a = 1$. Putting this back into either will give us $b = -6$.

23. In a monopoly market, the demand for a product is related to the price by the equation $p = -3x+36$, where x is the number of units demanded and sold each month and p is in hundreds of dollars. Revenue is given by price times the number of units sold. Profit is revenue minus cost. The monthly cost function is given by $C(x) = 2x^2+6x+10$. Find the price that maximizes profit.

    (a) \$2700 **(b)** \$300 **(c)** \$3500 **(d)** \$4700 **(e)** None of the above

    SOLUTION **(a)**: The revenue function is $R(x) = px = (-3x + 36)x = -3x^2 + 36x$. The Profit function is revenue minus cost, so it is $P(x) = R(x) - C(x) = -3x^2 + 36x - (2x^2 + 6x + 10) = -5x^2 + 30x - 10$. This is a parabola opening down. The maximum value for profit will happen at the vertex which will be at $x = \frac{-30}{-10} = 3$. The price that corresponds to $x = 3$ is $p = -3(3) + 36 = 27$. In hundreds of dollars this is \$2700.

24. An insurance company estimates that 40% of policyholders who have only an auto policy will renew next year and 60% of policyholders who have only a homeowners policy will renew next year. The company estimates that 80% of policyholders who have both an auto and homeowners policies will renew at least one of those policies next year.
Company records show that 65% of policyholders have an auto policy, 50% of policyholders have a homeowners policy, and 15% of policyholders have both an auto and homeowners policy.
Using the company's estimates, calculate the percentage of policyholders that will renew at least one policy next year.

(a) 20 **(b)** 29 **(c)** 41 **(d)** 53 **(e)** 70

SOLUTION **(d)**: According to the company's records, 65% of policyholders have an auto policy. This includes the 15% of policyholders who have both policies. That means 65% − 15% = 50% of policyholders have only auto policy.
The same way we find that 50% − 15% = 35% of policyholders have only homeowners policy.
The percentage of policyholders who will renew at least one policy next year will be: 40% of 50% plus 60% of 35% plus 80% of 15%, which is

$$0.4 \times 50 + 0.6 \times 35 + 0.8 \times 15 = 4 \times 5 + 6 \times 3.5 + 8 \times 1.5 = 20 + 21 + 12 = 53$$

25. Thirty items are arranged in a 6-by-5 array as shown:

$$\begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \\ A_6 & A_7 & A_8 & A_9 & A_{10} \\ A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{16} & A_{17} & A_{18} & A_{19} & A_{20} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{26} & A_{27} & A_{28} & A_{29} & A_{30} \end{array}$$

What is the number of ways to form a set of three distinct items such that no two selected items are in the same row or column?

(a) 200 **(b)** 760 **(c)** 1200 **(d)** 4560 **(e)** 7200

SOLUTION **(c)**: To make such selection, one has to first select three distinct rows, then select three distinct columns to select items from the three rows. While the order of the rows does not matter, the order of the columns is important, for example selecting the first entry from the first row and second entry from the second row is different from selecting the first entry from the second row and second entry from the first row. So the total number of ways is

$${}_6C_3 \times {}_5P_3 = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times (5 \times 4 \times 3) = 20 \times 60 = 1200$$